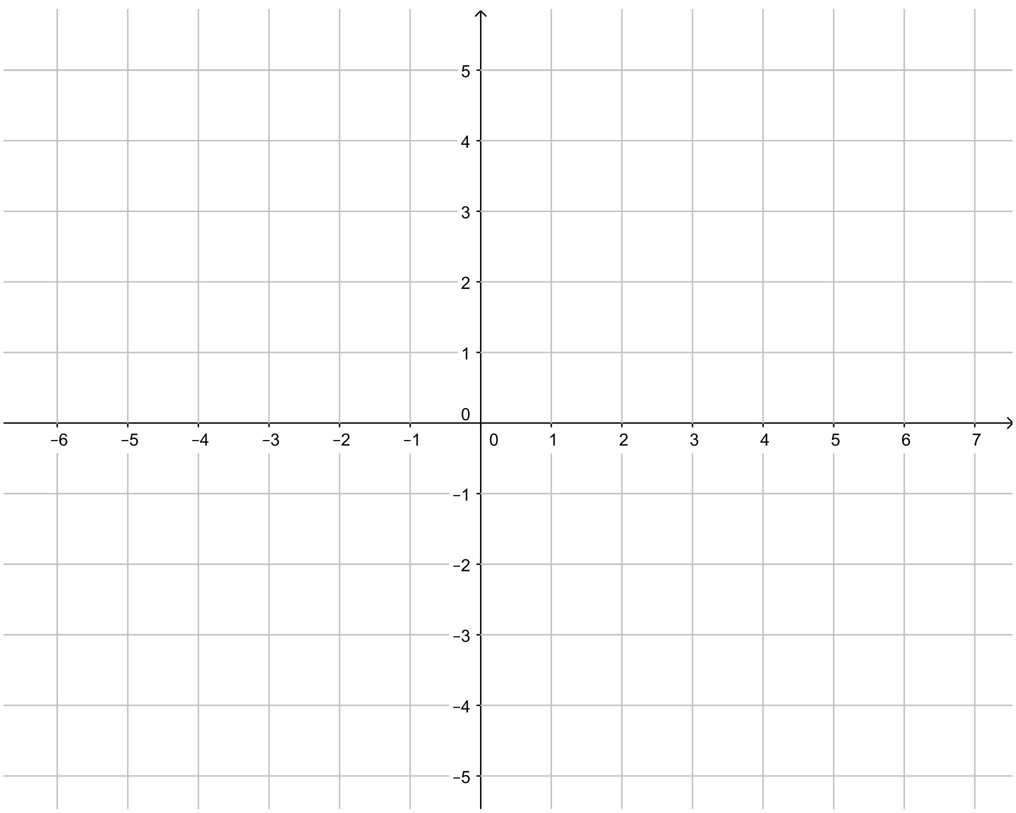
Limits and One-Sided Limits

Write the intuitive definition of a limit using proper notation.

Draw the graph of $h(x) = \dfrac{(5 - x^2)(x - 2)}{x - 2}$.



Use the graph and the intuitive defintion to find the following limits.

$\lim_{x\to 0} h(x) =$ ______________ $\lim_{x\to 2} h(x) =$ ______________ $\lim_{x\to -2} h(x) =$ ______________

One-Sided Limits

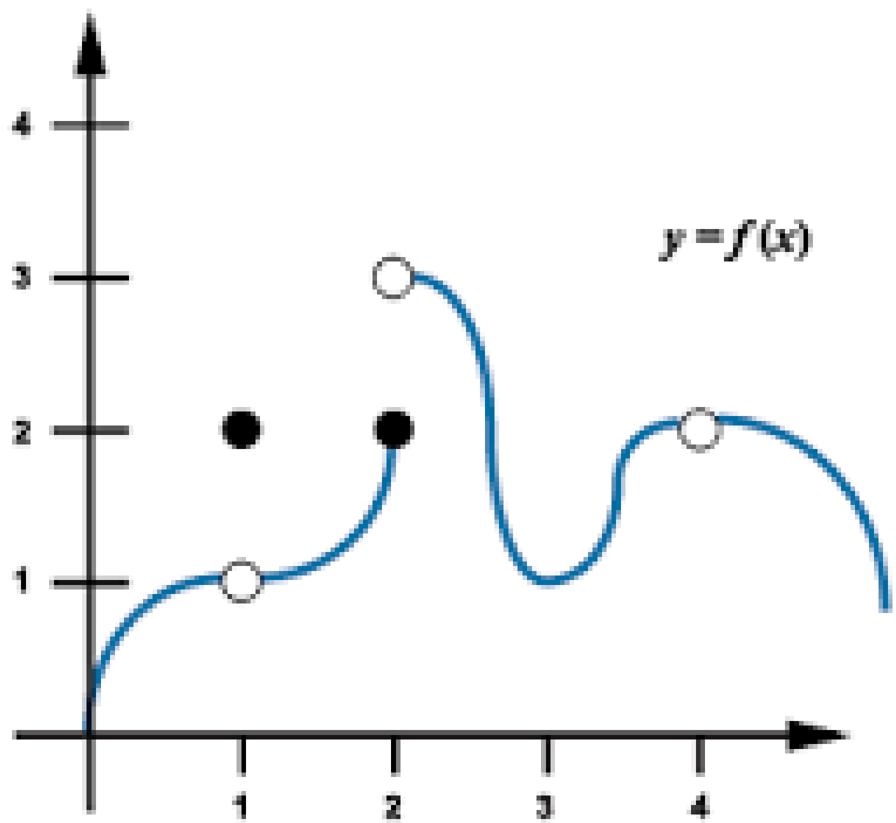
Intuitive Definition: $\lim_{x\to a^+} f(x) = L$ means $f(x)$ gets close to L as $x > a$ gets close to, but **not** equal to, a.

Alternate notation: $f(x) \to L$ as $x \to a^+$.

Similarly

Intuitive Definition: $\lim_{x\to a^-} f(x) = L$ means $f(x)$ gets close to L as $x < a$ gets close to, but **not** equal to, a.

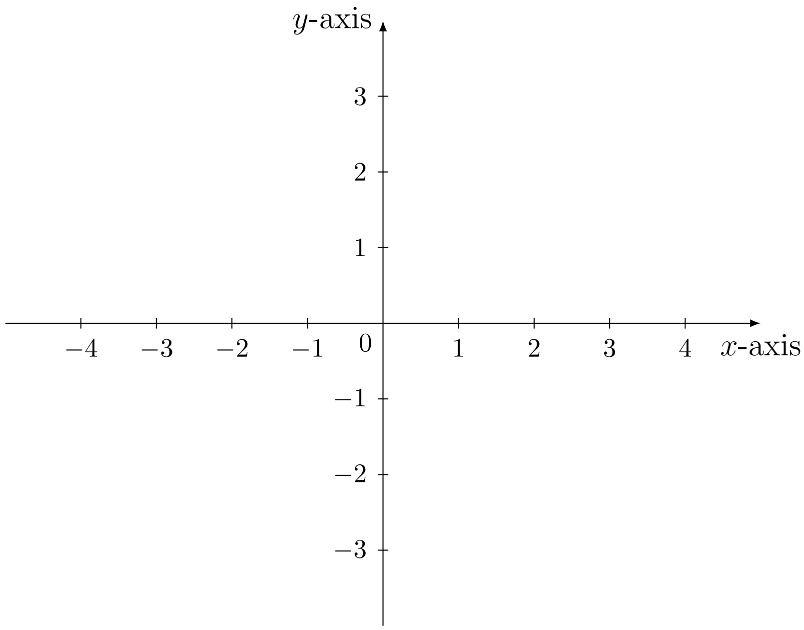
Alternate notation: $f(x) \to L$ as $x \to a^-$.



$\lim_{x\to 1^-} f(x) =$ ______________ $\lim_{x\to 1^+} f(x) =$ ______________

$\lim_{x\to 2^-} f(x) =$ ______________ $\lim_{x\to 2^+} f(x) =$ ______________

Draw the graph of $g(x) = \begin{cases} -1 & x < 0 \\ \dfrac{1}{x-1} & 0 \le x < 1 \text{ and } 1 < x < 3. \\ x - 1 & x \ge 3 \end{cases}$



Use the graph and the intuitive definitions to find the following limits. Write "DNE" if the limit does not exist, and "∞" if the outputs approach infinity.

$\lim_{x\to 0^-} g(x) =$ ________ $\lim_{x\to 0^+} g(x) =$ ________ $\lim_{x\to 0} g(x) =$ ________

$\lim_{x\to 1^-} g(x) =$ ________ $\lim_{x\to 1^+} g(x) =$ ________ $\lim_{x\to 1} g(x) =$ ________

$\lim_{x\to 3^-} g(x) =$ ________ $\lim_{x\to 3^+} g(x) =$ ________ $\lim_{x\to 3} g(x) =$ ________

How does your answer change if $g(x) = \begin{cases} -1 & x < 0 \\ \dfrac{1}{(x-1)^2} & 0 \le x < 1 \text{ and } 1 < x < 3? \\ x - 1 & x \ge 3 \end{cases}$

The One-Sided Limit Theorem

$\lim_{x \to a} f(x) = L$ iff[1] $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$.

Note: If the one-sided limits are different or do not exist (DNE), then the limit does not exist.

Find a function for which $\lim_{x \to 0} f(x)$ DNE.

Find a function for which $\lim_{x \to 0} f(x) = -\infty$.

<u>Intuitive Definition</u>: $\lim_{x \to a} f(x) = \infty$ if $f(x)$ gets as large as we want as x gets close to, but not equal to, a.

$\lim_{x \to a} f(x) = -\infty$ has a similar definition.

If the two one sided limits agree but are infinite ($\pm\infty$) then the limit is infinite.

Tabular Method

This method works to estimate $\lim_{x \to a} f(x)$ if we know the limit exists and we don't care to know how accurate the estimate is. The method is: create a sequence of values approaching a and inspect the corresponding outputs. Hopefully they get close to a single number - our estimate. You need a calculator for this method, so you won't be able to use it on a test, and it won't be good enough to satisfy "defend your answer" as I require. However, if you are good at metal arithmetic, it may allow you to check your answers.

Example: $\lim_{x \to 0} \frac{\sin(x)}{x}$. Use a calculator to complete the table if $f(x) = \frac{\sin(x)}{x}$. x is in radians.

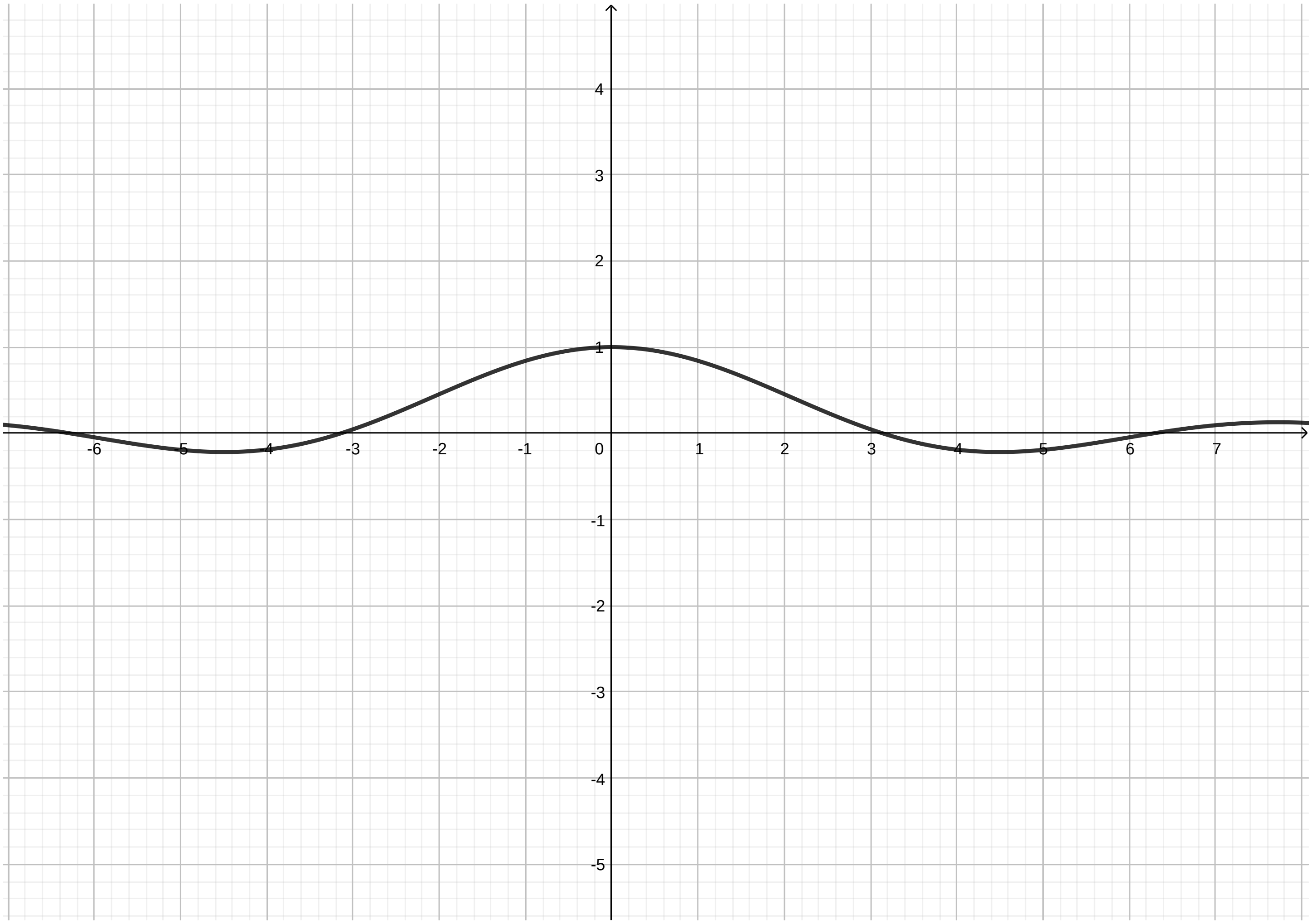
x	1	0.1	0.01	0.001
$f(x)$				

What does our limit appear to be? Notice that $f(x)$ is even[2] so that the sequence provides information about the inputs of $f(x)$ on both sides of 0.

[1] iff means "if and only if". One statement is equivalent to the other.

[2] $f(x)$ is even if $f(-x) = f(x)$ or, equivalently, the graph of $y = f(x)$ is symmetric about the y-axis.

Since we are using technology, we might as well graph $f(x)$ and see if our estimate appears accurate.



Let's try the tabular method for $g(x) = \sin\left(\frac{1}{x}\right)$ as $x \to 0^+$. Let $a_n = \frac{1}{n\pi}$. What is $g(a_n)$? Does this give a good estimate for the limit? Before answering, you may want to let $b_n = \frac{1}{(1/2 + 2n)\pi}$ and evaluate $g(b_n)$.

Here is the graph of $g(x)$: