Limits and Continuity

Intuitively, a function is continuous if its graph is "unbroken." Draw the graph of an arbitrary continuous function and show how to find a limit as x approaches any point a in its domain.

Definition: $f(x)$ is continuous at $x = a$ iff $\lim_{x \to a} f(x) = f(a)$

We can prove[1] x^n, for any positive integer n, x^r for any positive real number r, e^x, $\sin(x)$, and $\cos(x)$ are all continuous at any point in their domains.

Calculate $\lim_{x \to \pi/3} \sin(x)$.

Limits are linear operators (This is also true for derivatives and integrals.)

If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist then

1) $\lim_{x \to a} f(x) \pm g(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$, so limits distribute over addition and subtraction;

2) $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$ if k is a constant. So limits commute with constants.

Example: Calculate $\lim_{x \to \pi/6} 4\sin(x) - 6\cos(x)$.

[1] We would need to use infinite series so that we could define the functions in terms of x^n. See Apostol or Rudin.

Limits distribute over multiplication and division (This is **not** true for derivatives or integrals.)

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then

1)$\lim_{x\to a} f(x)\cdot g(x) = L\cdot M$;

2)$\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$ if $M \neq 0$.

Example: Calculate $\lim_{x\to 9} \frac{1}{9}x^2\sqrt{x} + \frac{e^9 x}{e^x}$.

Limits and composition

If $f(x)$ and $g(x)$ are continuous, then $(f\circ g)(x)$ is too, so

$$\lim_{x\to a} f(g(x)) = f(g(a)).$$

Example: Calculate $\lim_{x\to 2} \sin(e^{\ln(\pi x)})$

Example: Find $f(x)$ and $g(x)$ so that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ do not exist, but $\lim_{x \to a} f(x) + g(x)$ does exist.

We can further prove that inverses are also continuous. All these rules then tell us that polynomials, rational functions, $\ln(x)$, $\tan(x)$, $\cosh(x)$, $\sinh(x)$, and just about every other function you studied before Calculus is also continuous at any points in their domains.

<u>Rule of Thumb</u>: If $f(x)$ appears to be made from familiar functions, try substituting a in for x first to calculate $\lim_{x \to a} f(x)$.

Evaluate when possible. Otherwise write "indeterminate"[2] or "IND" for short.

$\lim_{x \to 2} \frac{5x - 12x}{\sqrt{x^3 + 4x}} =$ ______________ $\lim_{x \to 0^+} \frac{5x - 12x}{\sqrt{x^3 + 4x}} =$ ______________

$\lim_{x \to \pi} \frac{\sin(x - \pi/2)}{\cos(x)} =$ ______________ $\lim_{x \to 0} \frac{\sin(x)}{x} =$ ______________

$\lim_{x \to 0} \tan^{-1}(e^x) =$ ______________ $\lim_{x \to \ln(\pi/2)^-} \tan(e^x) =$ ______________

For what value of c is $g(x) = \begin{cases} 2x + 9x^{-1} & x \leq 3 \\ -4x + c & x > 3 \end{cases}$ continuous at $x = 3$?

[2] An indeterminate form is an expression involving two functions whose limit cannot be determined solely from the limits of the individual functions.