Introduction To Derivatives

(section 3.1)

Rigorous Definition The derivative of $f(x)$ at $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Alternate equivalent definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Notice the orginal definition is obtained from the alternate definition if we substitute h for $x - a$. $x - a$ is often called the **change in** x and is denoted Δx. The defintion then becomes

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

or

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

Alternate notation: $\left.\frac{df}{dx}\right|_{x=a}$.

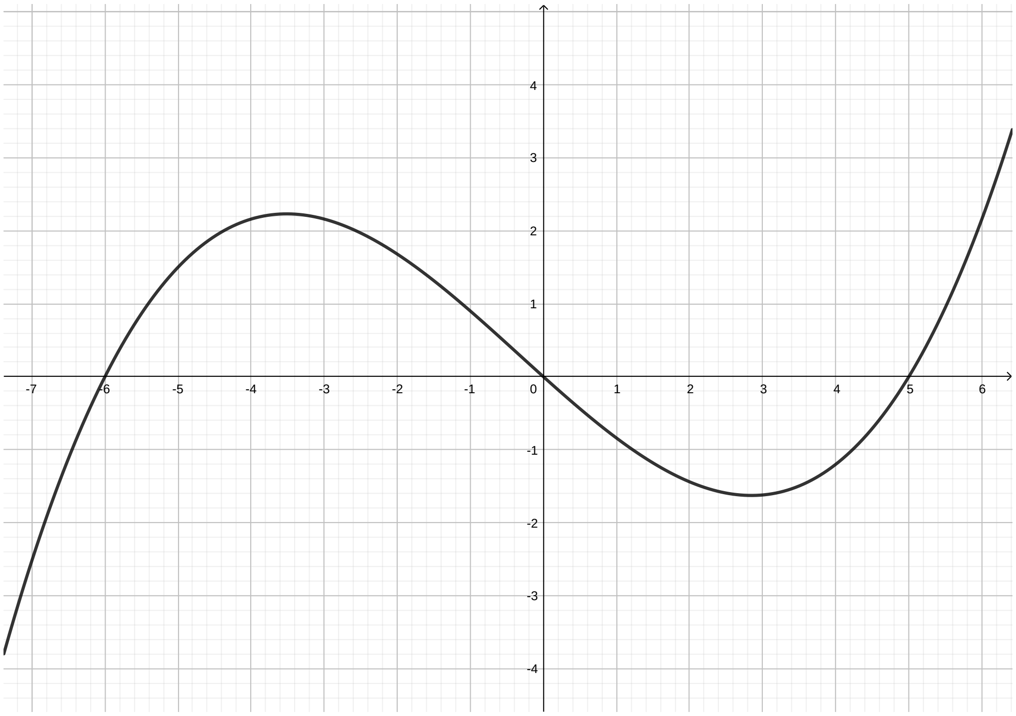
Intuitive Definitions

1) The slope of the tangent line at $x = a$.

2) The instataneous rate of change of $f(x)$ at $x = a$.

Draw a picture of secant lines approaching a tangent line to $y = f(x)$ at $x = a$.

Estimate $f'(x)$ for $x = -6, -4, 0, 3, 5$.



$f'(x)$ is a function. $\dfrac{df}{dx}$ is alternate notation. Draw an estimate of its graph on the axes above.

Frequently our tactics for finding $f'(a)$ are to find $f'(x)$ first and then substitute in a.

Example Use the definition to find $f'(x)$ if $f(x) = \sqrt{x}$. Then find $f'(1)$, $f'(4)$, and $f'(9)$. Do these outputs match with the graph of $f(x) = \sqrt{x}$?

Example Find the equation of the tangent line to the graph of $f(x) = \sqrt{x}$ at $x = 4$.

(Section 3.2)

Prove that if $f'(a)$ exists and is a real number, then $f(x)$ is continuous at a, but the converse statement is sometimes false. Recall that if $S \implies P$, the converse is $P \implies S$. In this case, I'm asking you to show that there exists a continuous function at a point that is not differentiable.

Limit rules allow us to create shortcuts for finding derivatives.

Power Rule If r is any **nonzero** number, then

$$\frac{d(x^r)}{dx} = rx^{r-1}$$

.

Examples Find the derivative of the following functions:

a) $f(x) = x^4$

b) $g(x) = x^{1/3}$

c) $h(x) = x^{-6}$

Presently we can prove this rule only if r is a positive integer using the binomial coefficient theorem. We will prove the rest later after we discuss the quotient rule and the derivatives of logarithms. Here is the positive integer proof.

What if $r = 0$? In other words, what is the derivative of $f(x) = 1$? Does your answer match the graph of $f(x)$? Generalize to any constant function $f(x) = k$.

Limits are linear operators and so too are derivatives.

Derivatives are Linear Operators

a) $(f + g)'(x) = f'(x) + g'(x)$

b) If k is a constant, then $(kf)'(x) = kf'(x)$.

Example If $g(x) = 3x^3 + 4x^2 + 3x - \frac{5}{x}$, find $\frac{dg}{dx}$, $\left.\frac{dg}{dx}\right|_{x=1}$, and $g'(-1)$.

In math 160 you will be told that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. This is, in fact, a good definition for e^x. If we define e^x this way and assume the derivative distributes over an infinite sum - not always true, but it does in this case - prove

$$\frac{de^x}{dx} = e^x$$

Example Find $\frac{dy}{dx}$ if $y = 3x^7 - 4x^\pi + 5e^x$.

As time allows: Prove the two linear operator properties.