Classifying Critical Points (4.3, 4.4)

If $f'(x) > 0$ for $x \in (a,b)$ and $a < x < y < b$, then the MVT implies that there is $c \in (x,y)$ so that

$$f(y) - f(x) = f'(c)(y - x) > 0.$$

This shows rigorously that $f(x)$ is increasing on (a,b) if $f'(x)$ is positive there. A similar proof shows $f(x)$ is decreasing on (a,b) if $f'(x) < 0$ there. This gives us a way to check if a critical point is a maximum or minimum.

First Derivative Test: If $c \in (a,b)$ is a critical point of $f(x)$ then

1) c is a maximum if $f'(x) > 0$ for $a < x < c$ and $f'(x) < 0$ for $c < x < b$.

2) c is a minimum if $f'(x) < 0$ for $a < x < c$ and $f'(x) > 0$ for $c < x < b$.

Draw two graphs illustrating the theorem.

Find and classify the critical points of $f(x) = x^4 - 2x^2 + 3$. "Classify" means to label as a local or global maximum or minimum.

Find and classify the critical points of $h(x) = \sqrt{3}\sin(x) + \cos(x)$ if $x \in [0, 2\pi]$. Where on $[0, 2\pi]$ is $h(x)$ increasing and decreasing?

The Mean Value Theorem can also be used to prove:

1) $f(x)$ is concave up on (a, b) if $f''(x) > 0$ on (a, b).

2) $f(x)$ is concave down on (a, b) if $f''(x) < 0$ on (a, b).

Draw a graph with added tangent lines to illustrate this.

This gives us another way to classify a critical point as a minimum or a maximum.

Second Derivative Test: If c is a critical point of $f(x)$ and

(1)$f''(c) < 0$, then c is a maximum.

(2)$f''(c) > 0$ then c is a minimum.

(3)$f''(c) = 0$ then nothing can be concluded.

Why can nothing be concluded in the third case? Draw graphs of $y = x^3$, $y = x^4$, and $y = -x^4$; then apply the second derivative test to the critical point at $x = 0$.

Definition: c is an **inflection point** of $f(x)$ if the graph of $f(x)$ switches concavity at $x = c$.

Find and classify the critical points for $g(x) = x^4 - 4x^3$ using the second derivative test when it applies. Where is the graph concave up and concave down? What are the inflection points?

Find and classify all critical points and find all inflection points of $k(x) = x^{1/3}(x+4)$. Where is $k(x)$ increasing and decreasing? Where is it concave up and concave down? Are there any vertical or horizontal asymptotes?