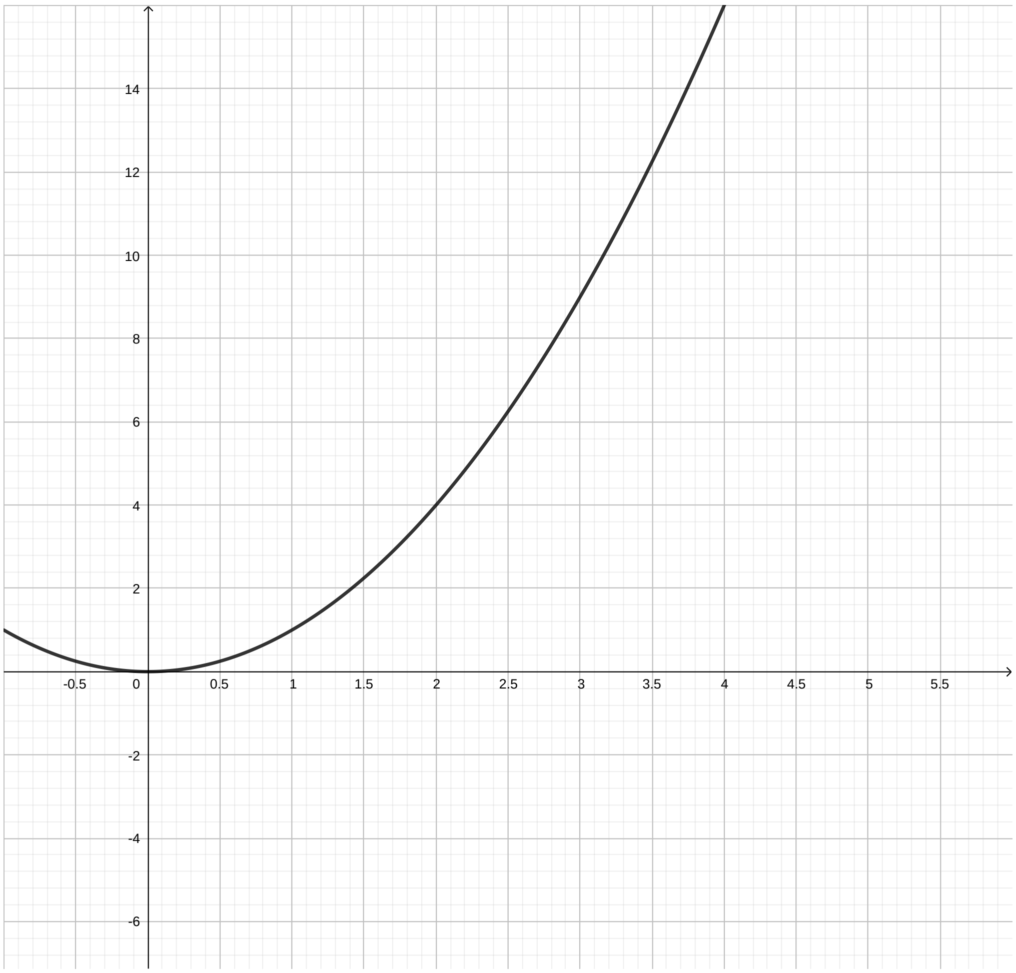
The Definite Integral (5.1 and 5.2)

How can we estimate the area under the curve over the interval $[0, 4]$?



Defintion: The **definite integral of $f(x)$ over** $[a, b]$ is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\frac{b-a}{n}$$

where x_i^* is any value in the ith subinterval[1].

$$\sum_{i=1}^{n} f(x_i^*)\frac{b-a}{n}$$

is a **Riemann sum**.

[1] The subintervals do not have to have equal length. We can use any n-subintervals for each Riemann sum as long as the maximum length of the subintervals approaches zero as n approaches infinity.

Use A Riemann sum with n = 4 and x_i^* equal to the midpoint of each subinterval to estimate $\int_0^8 x^2\,dx$.

Intuitively the definite integral is a "net area," where area above the x-axis is positive and area below the x-axis is negative.

Find the following integrals using graphs.

$$\int_{-\pi}^{\pi} \sin(x)\,dx$$

$$\int_0^6 \sqrt{36 - x^2}\,dx$$

Write an integral equal to

$$\lim_{n\to\infty} \frac{3}{n} \sum_{j=1}^{n} \left(2 + \frac{3j}{n}\right)^4.$$

Write a limit equal to

$$\int_{-3}^{1} \cos(x)\, dx.$$

Properties of Definite Integrals

Make a picture proof for the following properties 1) and 2).

1) If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0$.

2) If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

Prove property 3) using Riemann sums.

3) The definite integral is a linear operator. That means:

(a) $\int_a^b f(x) + g(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$; and

(b) $\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$ if c is a constant.

4) Use a picture proof to show $\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$.

5) Use Riemann sums to prove $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

6) Use a picture proof to show that $f(x) \leq g(x) \implies \displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

Evaluate the following integrals using properties of integrals and geometry.

1) $\int_{-1}^{2} 7\,dx$

2) $\int_{-5}^{5} \sin(3x^2)x^3 - \sqrt{25 - x^2}\,dx$

3) $\int_{2}^{-1} (1 - x)\, dx$

Prove $\frac{\pi}{12} \leq \int_{\pi/6}^{\pi/3} \cos(x)\, dx \leq \frac{\sqrt{3}\pi}{12}$.