Proof and Consequences of FTC (5.5, 5.6)

Let $F(x)$ be an antiderivative of $f(x)$. Write the Mean Value Theorem using F and f over the interval $[x_{i-1}, x_i]$. Use x_i^* instead of c in the statement. Don't forget the hypotheses of the theorem!

Using your result above, rewrite the Riemann sum of $\int_a^b f(x)\,dx$ with three subintervals in terms of $F(x)$.

Now prove the Fundamental Theorem of Calculus **assuming an antiderivative of the integrand exists**.

This shows a definite integral equals the **total change** in the antiderivative over the interval of integration.

If $s(t)$ is position, $v(t)$ velocity, and $a(t)$ is acceleration at time t, then

$$\int_a^t v(\tau)d\tau = s(t) - s(a) = \Delta s$$

and

$$\int_a^t a(\tau)d\tau = v(t) - v(a) = \Delta v.$$

Show graphically in each case how area is related to the integrand and to time. Explain how the units agree with the geometry.

A particle moves along a line so that $v(t) = 6t^2 - 6t - 36$ meters per second. Find the displacement of the particle over the time interval $0 \leq t \leq 4$.

How far did the particle travel during this time? For instance, a marathon runner on an out-and-back course runs 13 miles out and 13 miles back for a total distance of 26 miles but with a displacement of 0 miles. Draw a picture of the particle's motion.

The **marginal revenue** $R'(x)$ is the derivative of the revenue function $R(x)$, where x is the number of units sold. What does $\int_{1000}^{8000} R'(x)\,dx$ represent?

I proved the FTC assuming an antiderivative of the integrand exists. But does every integrand have an antiderivative? We can answer in the affirmative if definite integrals of that integrand exist. A new type of function is required; let's call it an **integral function**. It looks like $\int_a^x f(t)\,dt$ where a is a constant. The domain of this new function is all values of x for which the integral exists. If $g(x) = \int_a^x f(t)\,dt$, what is $g'(x)$? The FTC would tell us it is $f(x)$. How?

This, then, gives us a new derivative formula to memorize:

$$\frac{d \int_a^x f(t)\,dt}{dt} = f(x)$$

In words this formula says that an antiderivative of $f(x)$ is $\int_a^x f(t)\,dt$.

Let's practice using this formula. Find the derivatives of the following functions.

1) $g(x) = \int_0^x \cos(t^2)\,dt$

2) $h(y) = \int_y^2 x^7 - 4x^5\,dx$

3) $J(t) = \int_{t^2}^3 p^3 \arctan(p-2)\,dp$

4) What is the derivative of $K(x) = \int_x^{x^2} \tan(m)\,dm$?

But I haven't proved the FTC; I can't use the FTC in the proof - that is circular reasoning. I will have to use the definition of a derivative to prove the integral function is an antiderivative. The proof assumes the integrand is bounded in an interval about the point of differentiation, that is, there exists two functions, $m(h)$ and $M(h)$ so that $m(h) \leq f(x) \leq M(h)$ if $x - |h| \leq x \leq x + |h|$. The proof is below, something I will not test or quiz you about.