Density and Average Value (HW #1)

A **density** is a quantity with a ratio of units. For instance,

$\rho(x) = 3$ kg/m is a linear density for a chain with x equal to the position on the chain relative to an indexing axis.

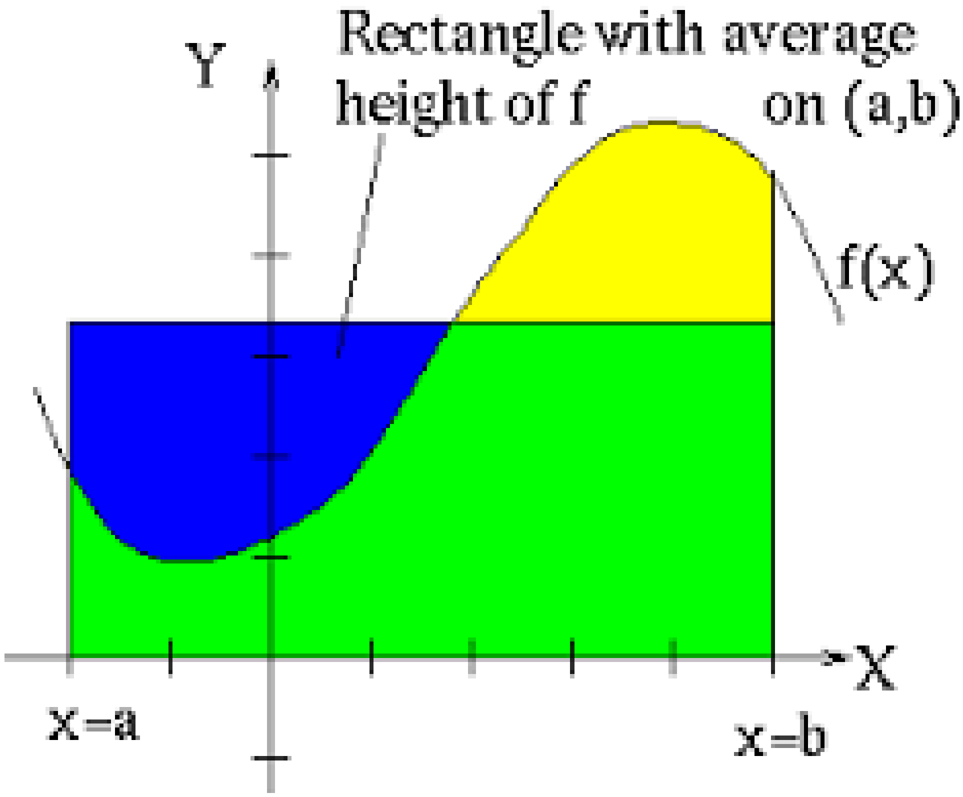
$D(r) = 40r$ humans/km^2 is a population density where r is the number of kilometers from a nuclear reactor.

$I(t) = 40\sin(t)$ Coulombs/sec is a density equal to the current in a circuit.

Integrating densities with respect to the unit in the denominator gives total change in terms of the upper unit.

Suppose the velocity of fluid particles through a tube with radius 5 cm is $v(r) = 10 - .3r - .4r^2$ cm/sec where r is the distance from the center of the tube. How many cm^3 of fluid flows through the portion of the tube for which $0 \leq r \leq 2$ each second?
Hint: the r-axis indexes small rings; be sure the units match!

The **average value of $f(x)$ over an interval** $[a, b]$ is the height of a rectangle with width $b - a$ that has the same signed area equal to the net area of $f(x)$ from a to b.



Use an integral to write the average value of $f(x)$ over $[a, b]$.

Find the average value of $\cos^2(x)$ over $[0, 2\pi]$.
Recall $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ so that $\cos(2x) = 2\cos^2(x) - 1$. Solve for $\cos^2(x)$ to obtain a useful trig identity.

The average value of a function and density functions combine to give the formula for the **center of mass**. It is a **weighted average** where the density of a material provides the weight. The formula for the center of mass of a metal rod of length L and with linear density $\delta(x)$ is

$$\bar{x} = \frac{\int_0^L x\delta(x)\,dx}{\int_0^L \delta(x)\,dx}.$$

This formula is often abbreviated as

$$\bar{x} = \frac{M_y}{M}$$

where M_y is the **moment about the y-axis** and M is the mass of the bar. We will talk about this idea in detail once we reach chapter eight.

You may not understand where the average value is in the formula for $\bar{x}$. We need to use the substitution technique to see it: Let $m(x)$ be an antiderivative of $\delta(x)$. Then Letting $u = m(x)$ implies

$$\bar{x} = \frac{\int_{m(0)}^{m(L)} x\,dm}{\int_{m(0)}^{m(L)} dm} = \frac{\int_{m(0)}^{m(L)} x\,dm}{m(L) - m(0)}$$

which is the average of the function $x(m)$ over the interval $[m(0), m(L)]$.