Partial Fractions (HW #3)

Compute $I = \int \frac{x}{x^2+3x+2}\,dx$. This is easy if we can find A and B so that

$$\frac{x}{x^2+3x+2} = \frac{x}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}.$$

If such an A and B can be found, what would we do next?

You may have seen the technique of finding A and B in precalculus; it is called partial fractions. Let's look at the traditional way it is taught and then find a way to speed up your work.

I hope you can go through this process much more quickly using the "cover-up method." I will perform the method as quickly as I want you to learn how to do it, and then show why it works.

Slower:

$$\left.\frac{x}{x+1}\right|_{x=-2} = A \implies A = 2 \qquad \text{and} \qquad \left.\frac{x}{x+2}\right|_{x=-1} = B \implies B = -1.$$

Here is why it works:

Let $f(x) = \dfrac{x}{(x+2)(x+1)} = \dfrac{A}{x+2} + \dfrac{B}{x+1}$. Then

$$(x+2)f(x) = A + \frac{B(x+2)}{(x+1)}$$

so that if $x = -2$ then

$$\frac{-2}{-2+1} = A.$$

Write out the similar reasoning for finding B.

Compute $\int \frac{2x+3}{x(x-1)(x-2)}dx$

Compute $\int \frac{-2x+4}{(x^2-9)(x-1)}dx$. What should we do if instead we have $\int \frac{-2x+4}{(x^2+9)(x-1)}dx$?

What if a factor is squared?

$$g(x) = \frac{x^2+2}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)^2} + \frac{C}{x+1}.$$

Perhaps you think it should be

$$g(x) = \frac{x^2+2}{x(x+1)^2} = \frac{A}{x} + \frac{C'x+B'}{(x+1)^2}.$$

You would be correct, but if we define C and B so that $C'x + B' = C(x+1) + B$, then

$$\frac{C'x+B'}{(x+1)^2} = \frac{C(x+1)+B}{(x+1)^2} = \frac{B}{(x+1)^2} + \frac{C}{x+1}.$$

Thus

$$g(x) = \frac{x^2+2}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)^2} + \frac{C}{x+1}.$$

implies

$$(x+1)^2 g(x) = \frac{x^2+2}{x} = \frac{A(x+1)^2}{x} + B + C(x+1)$$

so that if we **differentiate** and then substitute $x = -1$ we will be left with C on the right. Finish the process to find C and then compute $I = \int \frac{x^2+2}{x(x+1)^2}\,dx$.

Higer powers of a factor require higher order derivatives. I will go no higher than squares in this class.

Compute $\int \frac{x^2}{x^2+3x+2}\,dx$. Divide first so the degree on top becomes smaller than the degree in the denominator.