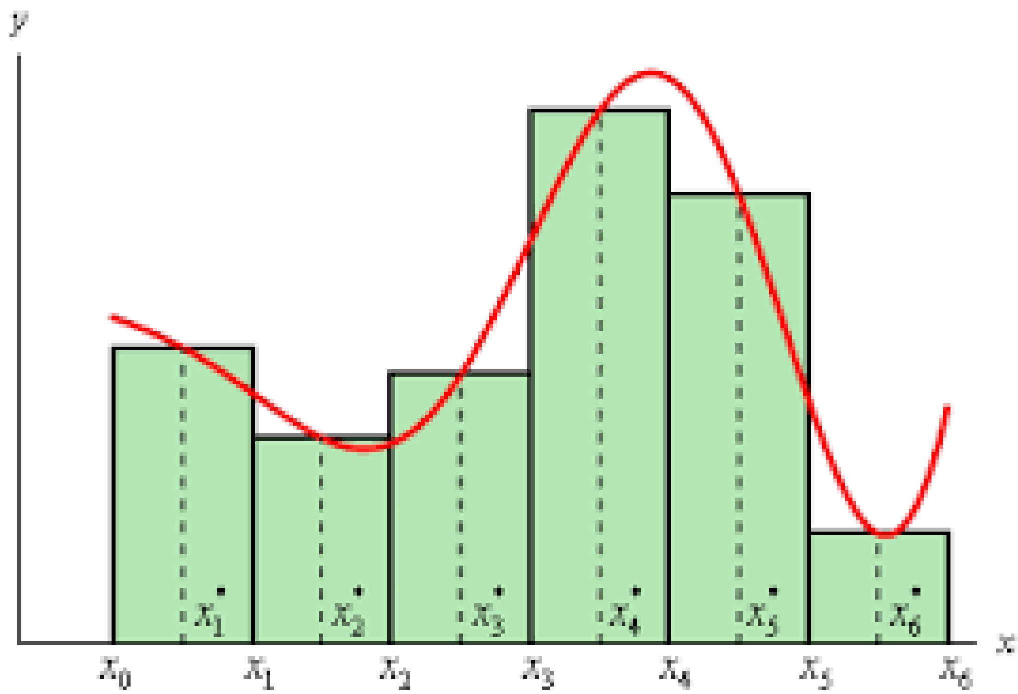
Simpson's Rule (HW #4)

The **definite integral of $f(x)$ over $[a, b]$** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\frac{b-a}{n}$$

where x_i^* is any value in the ith subinterval[1] We have been finding exact values of definite integrals when we are able to use geometry or find an antiderivative, but there are many cases when that cannot be done. Often a rule for a function does not exist. In that case we must estimate the definite integral.

A first method is to use the Riemann sum using the midpoints of n subintervals. This is denoted M_n.



$$\int_a^b f(x)\,dx \approx M_n = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i^*)$$

choosing x_i^* to be the midpoint of each subinterval.

Numerical Analysis is the branch of mathematics concerned with approximation using machines; it has proved the error for the midpoint method is

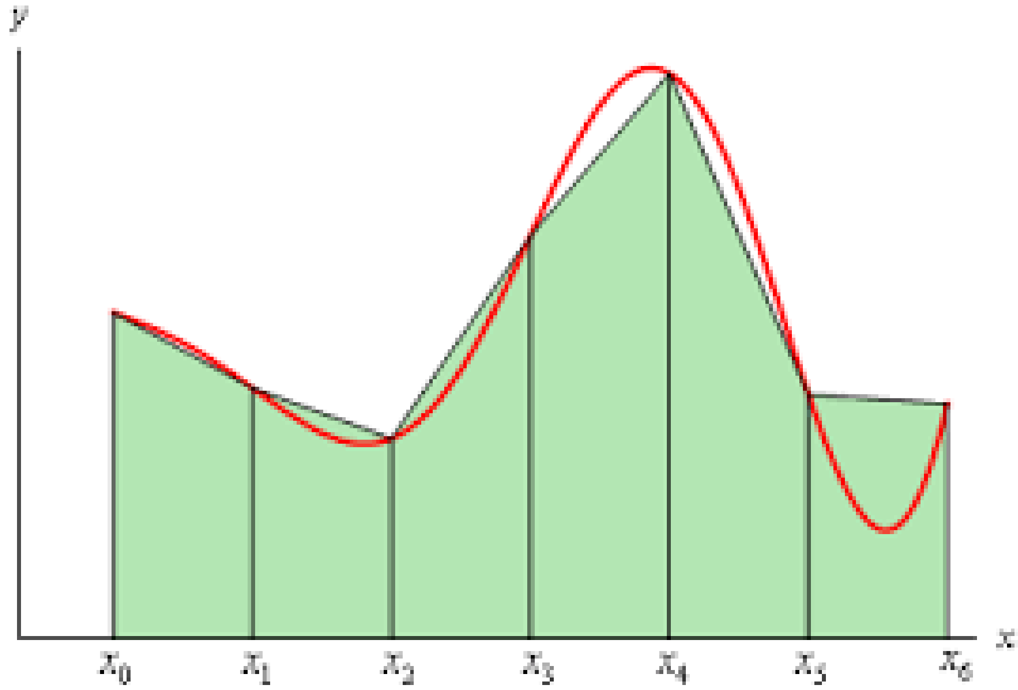
$$E(M_n) \le \frac{K(b-a)^3}{24n^2}$$

where

$$K = \max_{a\le x\le b} |f''(x)|.$$

[1]The subintervals do not have to have equal length.

Another method is the Trapezoidal method, since we know the area of a trapezoid for each subinterval is $\dfrac{(b-a)(f(x_i)+f(x_{i+1}))}{2}$. This approximation using n subintervals is denoted T_n.



It might look better, but its error, in general, is about the same as the midpoint rule:

$$\int_a^b f(x)\,dx \approx T_n = \frac{b-a}{2n}\left[f(x_0) + \left(2\sum_{i=1}^{n-1} f(x_i)\right) + f(x_n)\right]$$

where x_i is the ith endpoint of the subintervals, and the error is

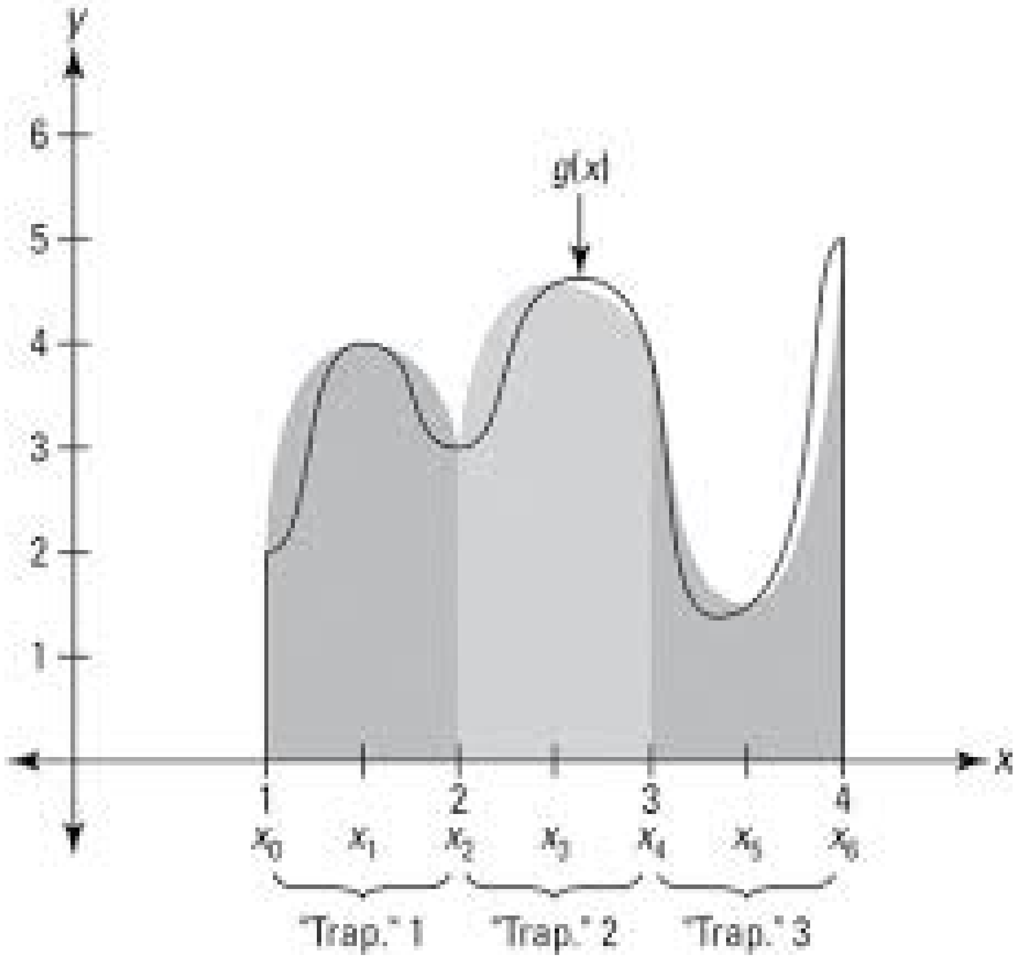
$$E(T_n) \leq \frac{K(b-a)^3}{12n^2}$$

where

$$K = \max_{a\leq x\leq b} |f''(x)|.$$

Why does the formula for T_n have the twos?

Industry uses **Simpson's** method. It fits parabolas using the endpoints of two subintervals. Hence we need an even number of subintervals.



We can show that the new formula is

$$\int_a^b f(x)\,dx \approx S_n = \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

The error is much smaller for large n than the Midpoint or Trapezoidal method:

$$E(S_n) \le \frac{K(b-a)^5}{180n^4}$$

where

$$K = \max_{a \le x \le b} |f^{(4)}(x)|.$$

You do not need to memorize any of these formulas for this class!!

Find n so that $E(S_n) < 0.01$ for $\int_0^1 e^{x^2}\, dx$. The smaller n is, the more money you save.

But how would we find the estimate for the integral? We would need to evaluate e^{x^2} at the endpoints of each subinterval - not easy to do by hand! We would use a calculator; the calculator estimates those values using a **Taylor Series** for e^{x^2}, something we will talk about in the near future.

One measure of heart health is the constant Cardiac Output which we denote by F with units of Liters/sec. Inject 6 mg of dye into the right atrium and measure the concentration, $C(t)$, in mg/Liter of dye coming out of the aorta every two seconds until the dye is gone. Then

$$6 = \int_0^{20} C(t) \cdot F \, dt$$

so that

$$F = \frac{6}{\int_0^{20} C(t) \, dt}.$$

Use Simpson's rule to estimate F if the following data has been collected:

t	C(t)	t	C(t)
0	0	10	4.3
2	4.1	12	2.5
4	8.9	14	1.2
6	8.5	16	0.2
8	6.7	18	0.0