Sequences and Geometric Series (HW #6)

Sections 10.1 and 10.2

A **sequence** is an ordered list of terms.

$\{a_n\}_{n=0}^{\infty} = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ can be written more briefly as $a_n = \frac{1}{2^n}$ where $a_0 = 1$, $a_1 = \frac{1}{2}$, $\ldots$.

The same sequence can be labeled differently: $b_n = \frac{1}{2^{n-1}}$ for $n = 1, 2, \ldots$.

We will be interested in finding $\lim_{n\to\infty} a_n$. The following theorem helps:

If $f(n) = a_n$ and $\lim_{x\to\infty} f(x)$ exists, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$.

So the rules for finding limits learned in math 150 apply in the same way to limits of sequences.

Find the following limits.

$\lim_{n\to\infty} \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{n^2}{e^n}$$

$$\lim_{n\to\infty} 10 - n^{-2}$$

$$\lim_{n\to\infty} (-1)^n$$

$\lim_{n\to\infty} S_n$ if $S_n = \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k$. This one is not so easy. We will need to shift, subtract, and solve for a new form for S_n.

$\lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k$ is called a **series**. This one is famous - it is called a **geometric series**. Instead of using the limit we write the series, as we did for improper integrals, with infinity as the upper limit of the sum:

$$\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k .$$

This is an example of a **convergent series** because the limit of its **partial sums** converges. The **nth partial sum** of this series is

$$S_n = \sum_{k=0}^{n} \left(\frac{3}{4}\right)^k .$$

A geometric series is of the form

$$S = \sum_{k=M}^{\infty} r^k$$

where M is any integer and r is any real number. We can use the shift, subtract, and solve method to find the behavior of any geometric series. Let's do that below.

So $\sum_{k=M}^{\infty} r^k$ diverges if $|r| \geq 1$ and converges if $|r| < 1$.

Unfortunately the shift, subtract, and solve method only works for geometric series. We will need other methods to determine convergence or divergence of other series.

Here is a basic test called the **Test for Divergence**:

If $S = \sum_{k=M}^{\infty} a_k$ and $\lim_{k\to\infty} a_k \neq 0$, then S diverges.

This test can only be used to prove a series diverges; when it does not apply, a series may or may not diverge.

Which of the following series does the Test for Divergence apply?

$$S = \sum_{k=-2}^{\infty} (1 + .9^k)$$

$$S = \sum_{k=1}^{\infty} \frac{1}{k}$$

$$S = \sum_{k=5}^{\infty} \frac{1}{k^2}$$

$$S = \sum_{k=0}^{\infty} (-1)^k$$

This page is only for those who worry about the rigor of my statements. It will not be covered in class or tested.

The rigorous definition for $\lim_{n\to\infty} a_n = L$ is for every ϵ there is an integer N so that if $n > N$, then $|a_n - L| < \epsilon$.

Then a proof of the Test for Divergence follows.

Let $S = \sum_{k=M}^{\infty} a_k$ be a series so that $\lim_{k\to\infty} a_k \neq 0$. Furthermore, let $S_n = \sum_{k=M}^{n} a_k$ be the nth partial sum of S. Let us assume that the theorem is sometimes false and that there is an S under these conditions that converges to a real number L. Then for every $\epsilon > 0$ there is an $N > 0$ so that if $n \geq N$ then $|S_n - L| < \epsilon$. But $\lim_{k\to\infty} a_k \neq 0$ implies there is a $k > N$ so that $|a_k| > 2\epsilon$. But then

$$|S_{k-1} - L| < \epsilon \implies |S_k - L| = |a_k + S_{k-1} - L| \geq |a_k| - |S_{k-1} - L| > 2\epsilon - \epsilon = \epsilon$$

This contradicts $|S_k - L| < \epsilon$, so no such S exists and the theorem must be true.