Tests for Convergence (HW #6)

Sections 10.3 and 10.4. Any time I ask if a series converges or diverges, you must defend your answer.

Does $S = \sum_{n=0}^{\infty} \frac{n}{3^n}$ converge or diverge? Hint: $n < 2^n$.

Comparison Test

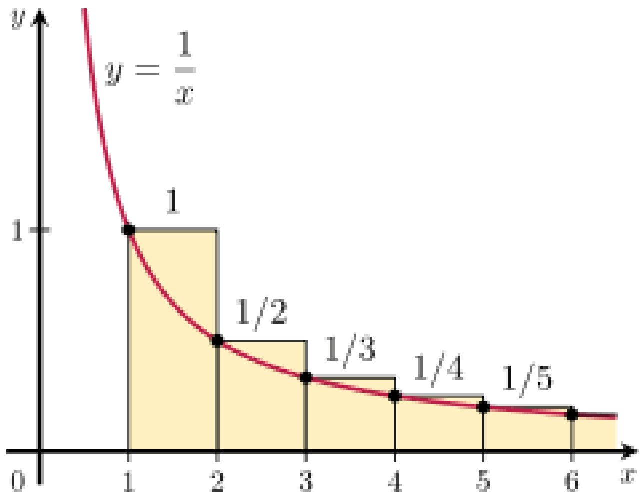
If there is a positive integer M so that $0 \leq a_n \leq b_n$ for $n \geq M$, then

1) $\sum_{n=L}^{\infty} a_n$ diverges implies $\sum_{n=L}^{\infty} b_n$ diverges; and

2) $\sum_{n=L}^{\infty} b_n$ converges implies $\sum_{n=L}^{\infty} a_n$ converges.

Comparison is easy, so we would like to know the behavior of more series that we can use for comparison with others. Harking back to improper integrals, we try to determine the behavior of **p-series**.

Does $S = \sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge? Hint: compare with the graph of $y = \frac{1}{x}$.



Integral Test

If $a_n = f(n)$, $f(x) > 0$, and $f'(x) < 0$ for $x \geq M$, then $\sum_{n=L}^{\infty} a_n$ has the same convergent or divergent behavior as $\int_M^{\infty} f(x)\,dx$.

Does $S = \sum_{n=L}^{\infty} \frac{1}{n^p}$ converge or diverge?

In summary we have the

P-Series

$\sum_{n=L}^{\infty} \frac{1}{n^p}$ converge if $p > 1$ and diverge if $p \leq 1$.

Does $S = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converge or diverge?

Does $S = \sum_{n=2}^{\infty} \frac{n}{n^3-2n-1}$ converge or diverge? Hint: Direct comparison with a p-series does not work - why? Instead, we use a "limit comparison."

Limit Comparison Test

If a_n and b_n are positive, and $\lim_{n\to\infty} \frac{a_n}{b_n} = k$, then

1) $k > 0$ implies $\sum_{n=L_1}^{\infty} a_n$ and $\sum_{n=L_2}^{\infty} b_n$ have the same convergent or divergent behavior.

2) $k = 0$ and $\sum_{n=L_1}^{\infty} b_n$ converges, then $\sum_{n=L_2}^{\infty} a_n$ converges. (We say that $\sum_{n=L_1}^{\infty} b_n$ dominates $\sum_{n=L_2}^{\infty} a_n$.)

3) $k = \infty$ and $\sum_{n=L_1}^{\infty} b_n$ diverges, then $\sum_{n=L_2}^{\infty} a_n$ diverges. (We say that $\sum_{n=L_2}^{\infty} a_n$ dominates $\sum_{n=L_1}^{\infty} b_n$.)

Does $S = \sum_{n=9}^{\infty} \frac{\sqrt{n}}{n+4}$ converge or diverge?

So far we have only presented tests for series with positive terms. The Taylor series for $\sin(x)$ and $\cos(x)$ have alternating signs. To handle such series, we have the

Alternating Series Test[1] (AST)

If $0 < a_{n+1} < a_n$ for $n \geq M$ and $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=L}^{\infty}(-1)^n a_n$ converges.

Does $S = \sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n)!}$ converge or diverge?

Does $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converge or diverge? This series is called the alternating harmonic series.

[1]See page 70 and 71 in the third edition of Walter Rudin's "Principles of Mathematical Analysis" for a proof.

Definition: $\sum_{n=L}^{\infty} a_n$ **Converges Absolutely** if and only if $\sum_{n=L}^{\infty} |a_n|$ converges.

Theorem: Absolute convergence implies convergence.[2]

Does $S = \sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \frac{1}{7^2} - \ldots$ converge absolutely? Defend your answer.

Does $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ converge absolutely?

A series that converges like the alternating harmonic series, but does not converge absolutely is said to **converge conditionally**.

[2]See page 71 and 72 in the third edition of Walter Rudin's "Principles of Mathematical Analysis" for a proof.