Ratio and Root Tests (HW #7)

Section 10.5 and 10.6.

Define a function $f(x) = \sum_{0}^{\infty} \frac{x^n}{n!}$. It's domain is all x for which the series converges. The Root test and the Ratio test will help us determine where such a series converges.

This function does indeed converge for any x. This is, in fact, e^x.

The **Ratio test** says that if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = r$, then $|a_n| \approx r^n$ in the tail end of the series. Why? Consequently $\sum_{n=L}^{\infty} a_n$ converges if $r < 1$ and diverges if $r > 1$. No conclusion is made if $r = 1$.

The **Root test** says that if $\lim_{n\to\infty} |a_n|^{1/n} = r$, then $|a_n| \approx r^n$ near the tail end of the series and so $\sum_{n=L}^{\infty} a_n$ converges if $r < 1$ and diverges if $r > 1$. No conclusion is made if $r = 1$.

We want to define e^2 to be equal to $S = \sum_{n=0}^{\infty} \frac{2^n}{n!}$. Does S converge? The ratio test works well.

Does $S = \sum_{n=2}^{\infty} \frac{n}{(\ln(n))^n}$ converge or diverge? Attempt to use the ratio test.

Now use the root test to prove $S = \sum_{n=0}^{\infty} \frac{n}{(\ln(n))^n}$ converges.

Does $S = \sum_{n=2}^{\infty} \left(\frac{3n+5}{5n+1} \right)^n$ converge or diverge?

For what values of x does the series $S = \sum_{n=2}^{\infty} \frac{x^n}{n!}$ converge?

Since $S = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all real x, we say the the **radius of convergence** of S is infinite. When a series represents a **power function**, an infinite polynomial in terms of $(x-a)^n$, we can often find a radius of convergence about a using the Ratio or Root tests. If the radius of convergence is R, then the interval $(a-R, a+R)$ is in the domain of the function. Once we check separately to see if the series converges at $x = a - R$ and $x = a + R$, then we get the **interval of convergence**, the domain of the function.

By the way, now that we have proved S converges everywhere, we are allowed to define $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. Integrating term-by-term, we see this definition shows that e^x is a solution to $y' = y$ for $y(0) = 1$.

Find the radius of convergence and interval of convergence for $f(x) = \sum_{n=0}^{\infty} \frac{(x-3)^n}{2n+1}$

Find the radius of convergence and interval of convergence for $g(x) = \sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n$

Find the radius of convergence and interval of convergence for $h(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$. $h(x)$ is equal to $\ln(x+1)$ on its domain; why can't the interval of convergence be larger?