Taylor Series (HW #7)

Section 10.8, 10.9.

What is $\left.\dfrac{d^k\, x^4}{dx^k}\right|_{x=0}$ for $k = 0, 1, 2, \ldots$?

Generalize to get an expression for $\left.\dfrac{d^k\, x^n}{dx^k}\right|_{x=0}$ for $k = 0, 1, 2, \ldots$.

Let $f(x) = 4 - 5x + x^2 - 3x^3 + 7x^4$ be a polynomial function of degree 4. What are $f^{(k)}(0)$ for $k = 0, 1, 2, \ldots$? Suppose we want to change $f(x)$ so that its fourth derivative at $x = 0$ equals 10. What should I change and why?

Generalize for $f(x) = \displaystyle\sum_{n=0}^{k} a_n x^n$. Suppose we want to change $f(x)$ so that its kth derivative at $x = 0$ equals 10. What should I change and why?

Differentiating and integrating polynomial functions is easy - can we express every function as a polynomial? YES! - over an open interval IF we allow a polynomial with an infinite amount of terms and the function has derivatives of all orders in that interval. This is motivation for the definition of a Taylor series for a function.

Definition: The Taylor series for a function $f(x)$ **expanded about $\mathbf{x = 0}$** is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

The Taylor Polynomial expanded about zero of degree k is the k^{th} partial sum of the Taylor series, $T_k(x)$. A Taylor series **expanded about 0** is a very important special type of Taylor series; it is given a special name, the **Maclaurin** series.

Taylor Series Theorem: $f(x) = T(x)$ on the interval of convergence of $T(x)$.

Find the Maclaurin series for e^x using the definition. **Memorize this series!!!**. What is the interval of convergence?

Find the Maclaurin series for $\cos(x)$ using the definition. **Memorize this series!!!**. What is the interval of convergence?

Find the Maclaurin series for $\sin(x)$ using the definition. **Memorize this series!!!**. What is the interval of convergence?

Find the Maclaurin series for $f(x) = (1 + x)^r$ using the definition. This is called the "binomial series."

Taylor polynomials are used to approximate the Taylor series and hence the function. There is an error formula[1] for Maclauren series that is similar to that for Simpson's rule:

$$|f(x) - T_n(x)| \leq \frac{K|x|^{n+1}}{(n+1)!}$$

Where $K \geq \max f^{(n+1)}(c)$ for all c between x and a.

Use the error formula to determine the smallest degree of the Taylor polynomial for e^x expanded about $x = 0$ that will estimate e^1 with error less than 10^{-14}. Hint: My calculator gives $\frac{3}{17!} = 8.4 \times 10^{-15}$ and $\frac{2}{16!} = 9.6 \times 10^{-14}$.

[1]I don't expect you to memorize it, but I do expect you to use it if the formula is given to you. A nice proof can be found on page 110 and 111 in "Principles of Mathematics," third edition, by Walter Rudin

Sometimes we want to expand a function about a different value of x. For instance, how do we find the Taylor series for $h(x) = \ln(x)$ expanded about $x = 1$? By translating the Maclaurin series. Do it.

Because

$$\left.\frac{d^k\,(x-a)^n}{dx^k}\right|_{x=a} = \begin{cases} 0 & n \neq k \\ n! & n = k \end{cases}$$

the Taylor Series of the function $f(x)$ **expanded about** $x = a$ is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The error formula becomes

$$|f(x) - T_n(x)| \leq \frac{K(x-a)^{n+1}}{(n+1)!}$$

if $K \geq \max f^{(n+1)}(c)$ for all c between x and a.

Substitution allows us to quickly write down some Taylor series. For instance, write the Taylor series expanded about $x = 0$ for e^{x^2} and $\cos(2x)$.

Now find a Maclaurin series for $e^{i\theta}$ using substitution. Combine like terms to prove

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Find some values of $e^{i\theta}$ for different values of θ.

The Taylor series for a function is unique, so if you can find it some other way, as we did using substitution, then you don't need to use the definition to check your answer. What is the Maclaurin series for $\frac{1}{1-x}$? Use it to find the Maclaurin series for $g(x) = \arctan(x)$. Hint: $\frac{1}{1-x}$ is a equal to a geometric series. Differentiate $g(x)$ and then use a substitution. Finally, integrate.