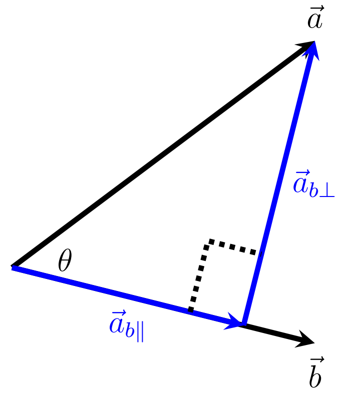
Dot Product (HW #8)

Every vector $\vec{a}$ can be decomposed with respect to any other vector $\vec{b}$ into a parallel component and a perpendicular component.

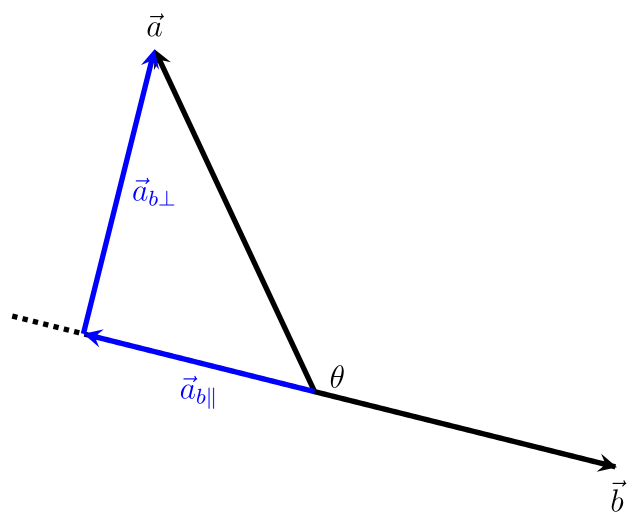


$\vec{a}_{b\|}$ is the **parallel component of $\vec{a}$ onto $\vec{b}$** and $\vec{a}_{b\perp}$ is the **perpendicular component of $\vec{a}$ to $\vec{b}$**. Each component has its own vector product. Notice $\vec{a} = \vec{a}_{b\|} + \vec{a}_{b\perp}$ and so if we know two of the three vectors, we can find the third.

The **dot product** of vectors $\vec{a}$ and $\vec{b}$ with angle θ between them is defined to be

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos(\theta).$$

Notice in the above diagram $\vec{a}_{b\|}$ is in the same direction as $\vec{b}$ so that $\vec{a} \cdot \vec{b}$ is positive. If $\vec{a}_{b\|}$ were in the opposite direction as in the diagram below, the dot product would be negative. **Why is this so?**



$\|\vec{a}_{b\|}\| = \|\vec{a}\||\cos(\theta)|$ and so the dot product depends only on $\vec{b}$ and $\vec{a}_{b\|}$.

Find $\hat{\imath} \cdot \hat{\jmath}$, $\hat{\jmath} \cdot \hat{k}$, $\hat{k} \cdot \hat{\imath}$, and then write a generalization in terms of $\vec{a}$ and $\vec{b}$.

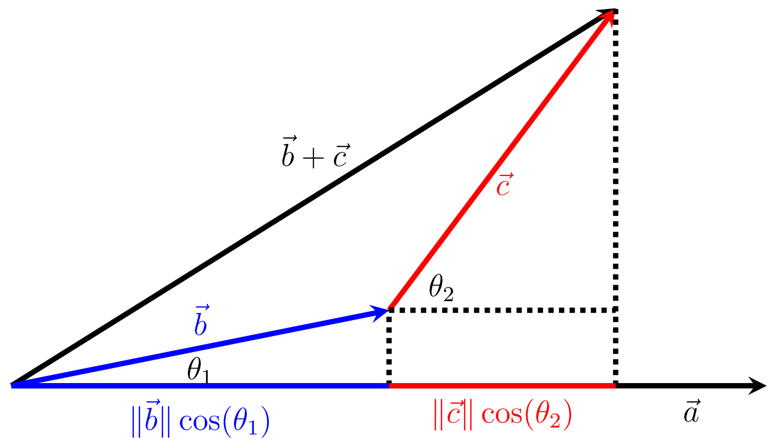
Find $\hat{\imath} \cdot \hat{\imath}$, $\hat{\jmath} \cdot \hat{\jmath}$, $\hat{k} \cdot \hat{k}$, and then write a generalization in terms of $\vec{a}$.

Prove $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

The dot product distributes over addition,

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}.$$

Here is a picture proof.



If θ is the angle between $\vec{a}$ and $\vec{b} + \vec{c}$, the picture $\implies$ $\|\vec{b} + \vec{c}\| \cos(\theta) = \|\vec{b}\| \cos(\theta_1) + \|\vec{c}\| \cos(\theta_2)$. Now finish the proof.

Use the distributive law to calculate $\langle 1, 2, -2 \rangle \cdot \langle 4, -1, 2 \rangle$.

We thus have an easy way to calculate dot products if given the component form of a vector. This works in any n-space and the formula is this:

$$\langle a_1, a_2, \ldots, a_n \rangle \cdot \langle b_1, b_2, \ldots, b_n \rangle = \sum_{i=1}^{n} a_i b_i$$

Use the dot product to prove $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$. Hint: use the two **unit** vectors with angles a and b. Assume without loss of generality that $a \geq b$.

Two Important Applications

The **Projection** of a vector $\vec{b}$ onto a vector $\vec{a}$ is $\vec{b}_{a\|}$. Sometimes the notation $\text{proj}_{\vec{a}}(\vec{b})$ is used where $\text{proj}_{\vec{a}}$ is the name of the function and $\vec{b}$ is the input. Projection is used to find best-fitting curves and Fourier Series, and $\vec{b}_{a\perp}$ is often called the **error** vector for these estimates.

Prove

$$\vec{b}_{a\|} = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}.$$

Use the formula from the last example to find the projection of $\vec{b} = \hat{\imath} + 2\hat{\jmath} - 2\hat{k}$ onto $\vec{a} = \langle 1, 1, -2 \rangle$. Then find $\vec{b}_{a\perp}$ and $\cos(\theta)$ where θ is the angle between $\vec{b}$ and $\vec{a}$.

The second application is **Work**. This is a concept in Physics and will be useful in multivariable calculus when we discuss line integrals. Earlier in this course you learned in one-dimensional problems

$$\text{Work} = \text{Force} \cdot \text{Displacement}$$

where the constant Force was parallel to the direction of the straight-line displacement. If the force was in the same direction as displacement, work was positive; if opposite, the work was negative. For now, our Force and Displacement will remain constant, but they will be vectors in n-space. Only the parallel component of the Force onto the Displacement vector will perform work, so the formula becomes

$$W = \vec{F} \cdot \vec{D}.$$

Find the work done by a force $\vec{F} = \langle 1, 2, 3 \rangle$ Newtons on a particle that has been displaced $\vec{D} = \langle -2, 5, 2 \rangle$ meters. The units of work are Joules in this case, or Newton $\cdot$ Meters.