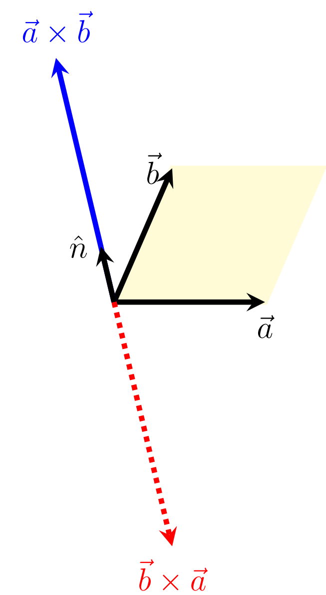
Cross Product (HW #8)

$$\vec{a} \times \vec{b} = \|\vec{a}\|\|\vec{b}\| \sin(\theta)\hat{n}$$

where $0 \leq \theta \leq \pi$ is the angle between $\vec{a}$ and $\vec{b}$, and $\hat{n}$ is a **unit** vector **perpendicular** to the plane spanned by $\vec{a}$ and $\vec{b}$ determined by the right-hand rule. The right-hand rule uses the right hand with fingers curling from $\vec{a}$ to $\vec{b}$ and thumb pointing in the direction of $\hat{n}$.



$\vec{a} \times \vec{b}$ is a **vector** while $\vec{a} \cdot \vec{b}$ is a scalar. The magnitude $\|\vec{a} \times \vec{b}\|$ equals the area of the parallelogram, and the direction is perpendicular to the plane spanned by $\vec{a}$ and $\vec{b}$.

Draw $\vec{b}_{a\perp}$ on the picture and notice how it is the height of the parallelogram. If $\vec{a}$ represents a wrench and $\vec{b}$ a force applied to the end of the wrench, then $\vec{b}_{a\perp}$ is the component of $\vec{b}$ that turns the wrench. The area of the parallelogram is a good measure of the turning force (or **torque**) since it increases with the length of the wrench and the length of $\vec{b}_{a\perp}$. The force would tighten a bolt with right-hand threads pointing in the direction of $\vec{a} \times \vec{b}$.

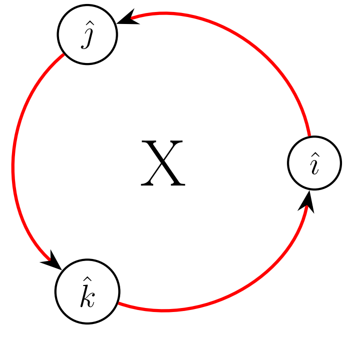
What factors of $\|\vec{a}\|\|\vec{b}\| \sin(\theta)\hat{n}$ multiply to equal the length of $\vec{b}_{a\perp}$?

Notice the right-hand rule guarantees the cross product is **not** commutative; rather, it is **anticommutative**:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

Find $\hat{\imath} \times \hat{\jmath}$, $\hat{\jmath} \times \hat{k}$, $\hat{k} \times \hat{\imath}$ using $\vec{a} \times \vec{b} = \|\vec{a}\|\|\vec{b}\| \sin(\theta)\hat{n}$.

Here is a nice summary of what we did; you need to add a negative to the answer if we multiply opposite the direction of the arrows.



Prove $(\lambda\vec{a}) \times \vec{b} = \lambda(\vec{a} \times \vec{b})$ if $\lambda \geq 0$. How does the proof change if $\lambda < 0$?

Find $\hat{\imath} \times \hat{\imath}$, $\hat{\jmath} \times \hat{\jmath}$, $\hat{k} \times \hat{k}$, and then write a generalization in terms of $\vec{a}$.

Fact: The cross product distributes over addition:

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}.$$

You can find the proof in **Second Year Calculus**, by David M. Bressoud. We assume it is true now and use it to find an easy way to calculate cross products if the vectors are given in component form.

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\vec{a} \times \vec{b} = (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$$

$$= (\qquad\qquad\qquad)\hat{\imath} + (\qquad\qquad\qquad)\hat{\jmath} + (\qquad\qquad\qquad)\hat{k}$$

$$= \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix}\hat{\imath} - \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix}\hat{\jmath} + \begin{vmatrix} \quad & \quad \\ \quad & \quad \end{vmatrix}\hat{k}$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

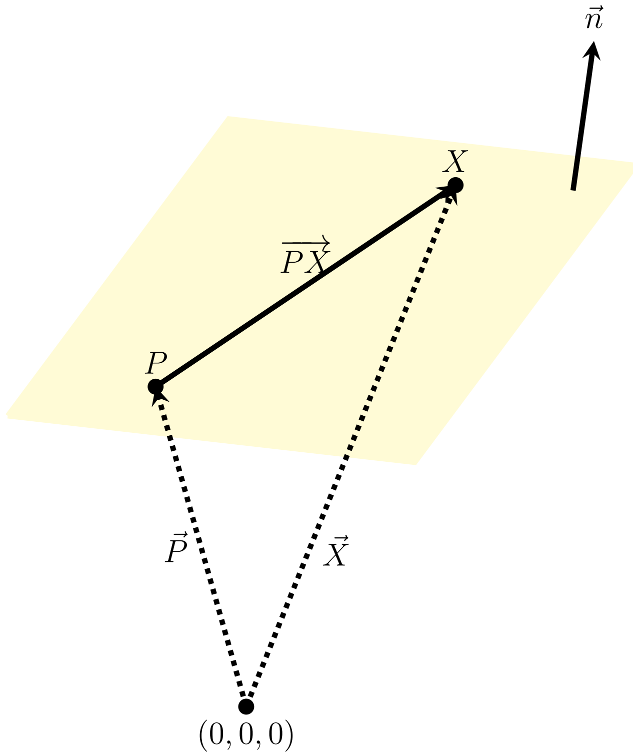
Find the area of the parallelogram spanned by $\vec{a} = \langle 1, 1, 2 \rangle$ and $\vec{b} = \langle 1, 3, 1 \rangle$. What is the area of the triangle spanned by $\vec{a}$ and $\vec{b}$?

Planes

Given a point P in a plane and a vector $\vec{n} = \langle a, b, c \rangle$ perpendicular to the plane we will derive the standard form for the equation of a plane:

$$ax + by + cz = d.$$

To do this, let $X = (x, y, z)$ be an arbitrary point in the plane and draw the vector $\overrightarrow{PX}$ in the plane. $\overrightarrow{PX}$ is perpendicular to $\vec{n}$.



But then

$$\overrightarrow{PX} \cdot \vec{n} = 0$$

$$\implies (\vec{X} - \vec{P}) \cdot \vec{n} = 0$$

$$\implies \vec{X} \cdot \vec{n} = \vec{P} \cdot \vec{n}$$

$$\implies \langle x, y, z \rangle \cdot \langle a, b, c \rangle = d$$

where $d = \vec{P} \cdot \vec{n}$. Multiplying gives the standard form of the plane

$$ax + by + cz = d.$$

Find an equation for the plane in standard form that contains the point $P = (2, 5, 7)$ and is parallel to the vectors $\vec{v} = \langle 2, 3, -1 \rangle$ and $\vec{w} = \langle 1, 2, 1 \rangle$.

Find an equation for the plane in standard form that contains the three points $P = (2, 5, 7)$, $Q = (1, -1, -2)$, and $R = (1, 1, 3)$.

The order of operations for vector arithmetic remains the same as real numbers since arithmetic is done component-by-component. Sometimes one order is not defined and the other order is; in that case, it is understood that the defined order is used.

If $\vec{v} = \langle 4, 1, 2 \rangle$, $\vec{a} = \langle 1, 1, 2 \rangle$, and $\vec{b} = \langle 1, 3, 1 \rangle$, simplify

$$5\vec{a} \times \vec{v} \cdot (2\vec{a} - 4\vec{b}) + \|\vec{v} - 2\vec{a}\|$$

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