Parametric Equations (HW #9)

Sections 11.1 and 11.2.

A position function with trace equal to a curve is a **parameterization** for the curve. The coordinate equations of this position function are the **parametric equations** for the curve.

Find the parametric equations for the line through the point $(1, -2)$ in the direction of the vector $\langle 2, 1 \rangle$. Then find the equation of the line in slope-intercept form and notice how the slope relates to the direction vector.

Find the parametric equations for the line through the point $(1, -2, 3)$ in the direction of the vector $\langle 2, 1, -1 \rangle$. There is no single equation describing this line in 3-space, but we can solve for t in each parametric equation to get "symmetric form."

To parameterize a circle of unit radius with center at $(0, 0)$ we need only find equations for x and y that satisfy the constraint $x^2 + y^2 = 1$. Explain how the trace is different for each choice below.

$x = \cos(t)$ and $y = \sin(t)$ for $0 \leq t \leq 2\pi$.

$x = \cos(2t)$ and $y = \sin(2t)$ for $0 \leq t \leq 2\pi$.

$x = \sin(t)$ and $y = \cos(t)$ for $0 \leq t \leq 2\pi$.

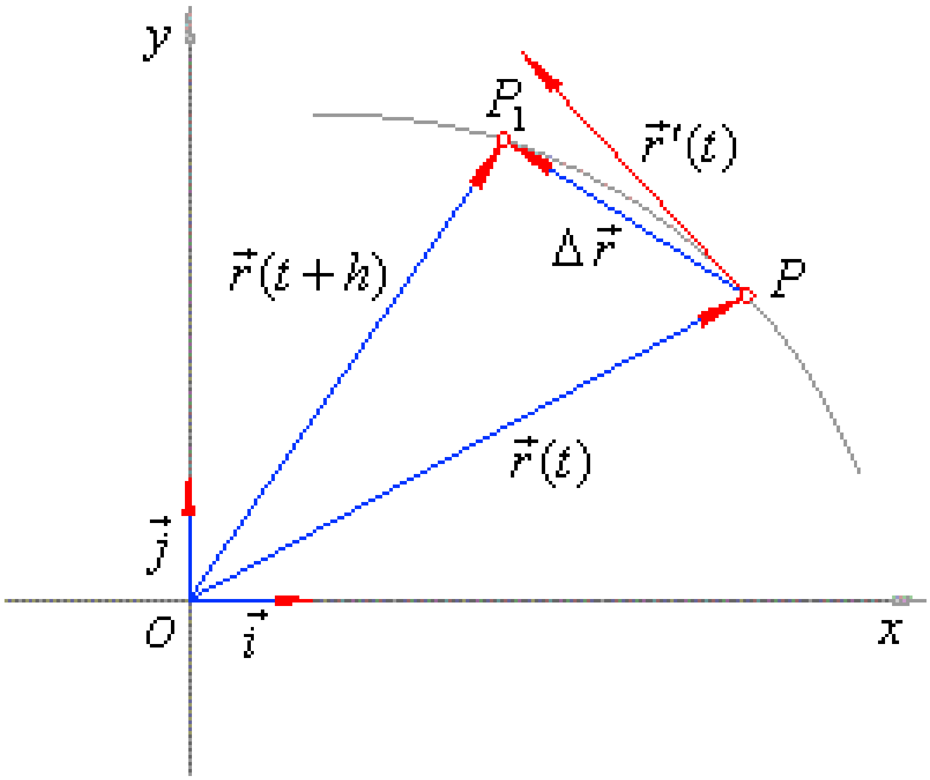
$x = x$ and $y = \sqrt{1 - x^2}$ for $-1 \leq x \leq 1$.

Find parametric equations for the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Find parametric equations for the hyperbola $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

Find parametric equations for any function $y = f(x)$. Give a particular example.

We differentiate a position function $\vec{r}(t)$ component-by-component the same way we add, subtract, or scalar multiply vectors. The derivative is a vector that lies on the tangent line to the curve pointing in the direction of orientation.



$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

$$= \left\langle \lim_{h\to 0} \frac{x(t+h) - x(t)}{h}, \lim_{h\to 0} \frac{y(t+h) - y(t)}{h} \right\rangle$$

$$= \langle x'(t), y'(t) \rangle.$$

$\vec{r}\,'(t)$ is the **velocity** of a particle that moves on the trace with parameterization $\vec{r}(t)$ and $\|\vec{r}\,'(t)\|$ is the **speed** of the particle.

$\vec{p}(t) = \langle 2\cos(5t), 3\sin(5t)\rangle$ is a position function for a particle. What are the parametric equations? What is the trace? What is the speed and velocity at $t = \pi$?

If $y = y(x)$ then the chain rule tells us that

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{y'(t)}{x'(t)}$$

when $x'(t) \neq 0$. Notice then that the velocity is horizontal when $y'(x) = 0$ and $y'(x)$ does not exist when the velocity is vertical: $y'(t) \neq 0$, but $x'(t) = 0$. Apply this to the elliptical example. Where is the velocity horizontal and were is it vertical if $0 \leq t \leq 2\pi$?

Recall that if s is the arc length, then $ds^2 = dx^2 + dy^2$. Dividing by $(dt)^2$ and taking a square root gives the speed:

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

But then when substituting into an integral, we can use

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2}\ dt.$$

Notice that this is the exact same formula we had for test 2 if $x = t$.

So we can calculate the arc length $s = \displaystyle\int_C ds$ of a curve C with position function $\vec{r}(t) = \langle x(t), y(t) \rangle$ for $a \le t \le b$ by

$$s = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\ dt.$$

Find the length of the curve with parametric equations $x = t^2 + 1$ and $y = t^2 - 1$ for $0 \le t \le 2$.

Find the surface area of the curve with parametric equations $x = t^2 + 1$ and $y = t^2 - 1$ for $1 \le t \le 2$ that is rotated about the x-axis. Notice y is non-negative with these restrictions on t.