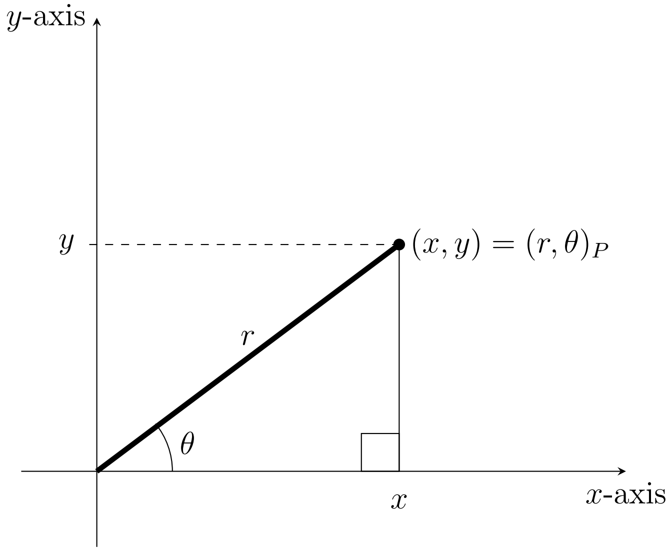
Polar Coordinates (HW #9)

Sections 11.3 and 11.4.



How can x and y be written in terms of r and θ? How can θ and r be written in terms of x and y?

Convert $(-1, -\sqrt{3})$ to polar coordinates. Remember the range of arctan.

Convert $\left(4, -\frac{\pi}{4}\right)_P$ to Cartesian coordinates (also known as rectangular coordinates.)

What are the **polar coordinate curves** in the Cartesian plane? That is, what curve is $r = k$ where k is a constant and $\theta = k$ in the xy-plane? In this class, r can be negative, but in math 200 I only use non-negative r. How will $\theta = k$ change if r is non-negative?

Sketch the polar curve $r = \theta$ in the xy-plane by plotting points.

Sketch the polar curve $r = 2\cos(\theta)$ in the xy-plane. It is perhaps quicker to convert the curve to a Cartesian equivalent. Another way to speed up the process is to notice that $\cos(-\theta) = \cos(\theta)$, so the graph is symmetric about the x-axis.

How does this last curve compare with $r = 2\cos(\theta - \pi/6)$? What does a translation of θ by $\pi/6$ do to the graph?

The cosine definition of a dot product can be used to convert the line $Ax + By = C$ to a polar equation given that the vector $\langle A, B\rangle \neq \langle 0, 0\rangle$. Let the angle between $\langle A, B\rangle$ and the polar axis be α and the usual polar representations $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

$$C = \langle A, B\rangle \cdot \langle x, y\rangle = \sqrt{A^2 + B^2}\sqrt{r^2\cos^2(\theta) + r^2\sin^2(\theta)}\cos(\theta - \alpha)$$

so

$$\frac{C}{\sqrt{A^2 + B^2}} = r\cos(\theta - \alpha).$$

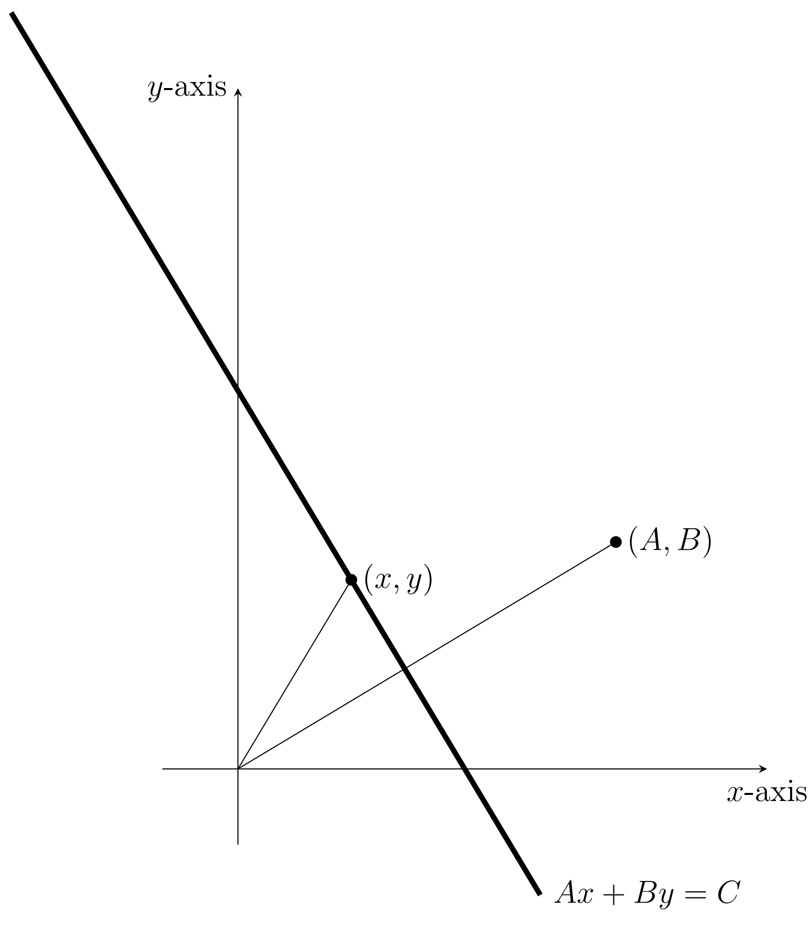
If we let the constant on the left be d, then we have

$$r\cos(\theta - \alpha) = d$$

or, if we want $r = f(\theta)$ form as we often do,

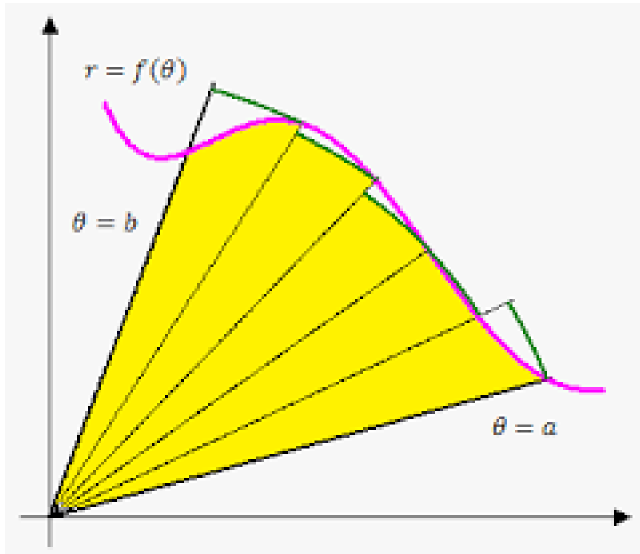
$$r = d\sec(\theta - \alpha).$$

Place θ, α, $\theta - \alpha$, and d on the following graph. Also place a right angle. Hint: At what point on the line is r smallest? What is the magnitude of the smallest r?

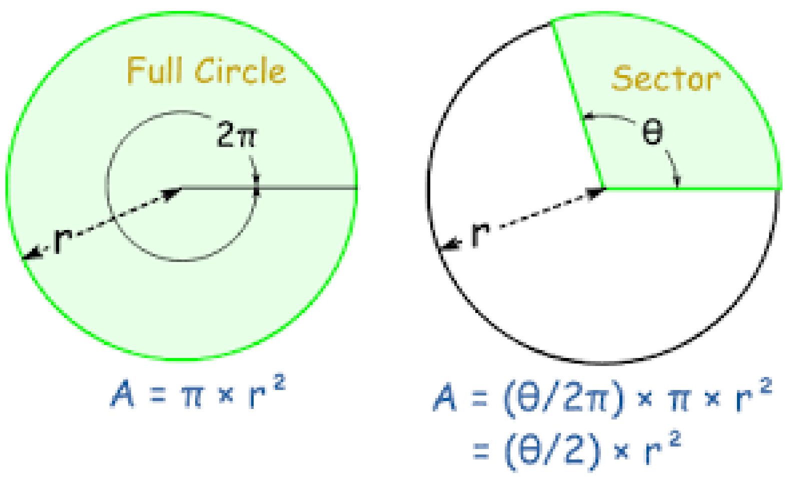


What does the line look like if $\alpha = 0$ or $\alpha = \pi/2$? Verify using both the picture and the algebra.

We estimate the area bounded by a polar function $r = f(\theta)$ for $a \le \theta \le b$ using the sum of areas of small sectors of circles.



The area of a sector of a circle with angle $\Delta\theta$ is $\dfrac{\Delta\theta\, r^2}{2}$.



As $\Delta\theta$ approaches 0, the estimate becomes exact and equals an integral:

$$\text{Area} \ = \int_a^b \frac{r^2}{2}\, d\theta$$

$$= \frac{1}{2}\int_a^b f^2(\theta)\, d\theta.$$

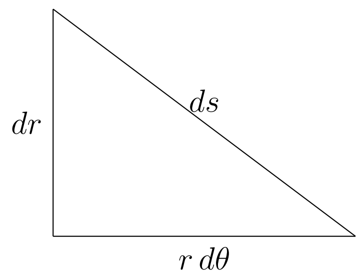
Sketch the curve $r = 2\cos(\theta) - 1$ remembering symmetry and then find the area between the inner loop and the outer loop. Hint: (Area of outer loop) - (Area of inner loop), and use symmetry.
(Answer: $\pi + 3\sqrt{3}$)

Arc Length

As with areas of sectors of circles, the arc length of a sector is proportional to the circumference:

$$\text{arc length} \quad = \frac{\Delta\theta}{2\pi} \cdot 2\pi r = \Delta\theta\, r.$$

Then as $\Delta\theta$ becomes very small, we have a figure that becomes closer and closer to a right triangle:



so that

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2.$$

Now factor $(d\theta)^2$ and apply a square root to get a formula for substitution in an integral.

$$ds = \sqrt{(r'(\theta))^2 + r^2}\, d\theta.$$

Compute the arc length of $r = \theta$ for $0 \le \theta \le \pi$. Hint: the anti-derivative of $\sec^3(x)$ is $0.5(\ln|\sec(x) + \tan(x)| + \sec(x)\tan(x))$.