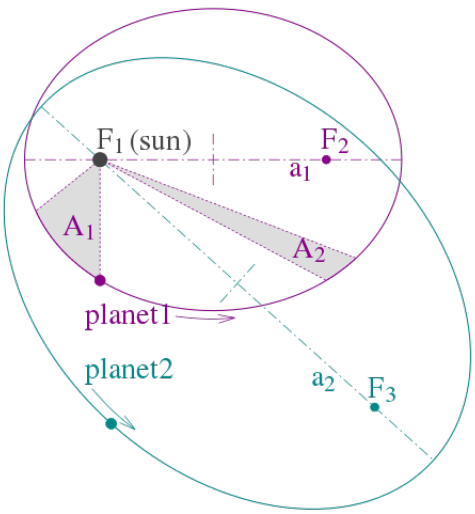
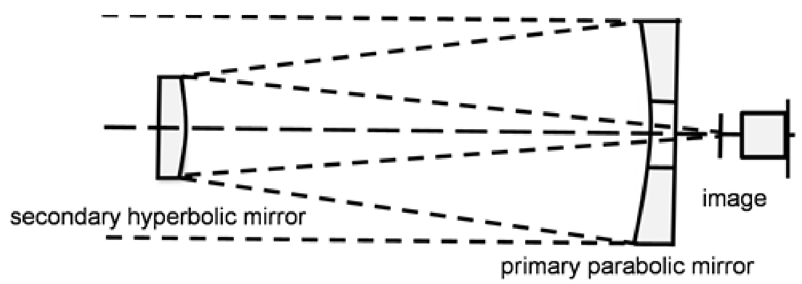
Conic Sections (HW #9)

Section 11.5.

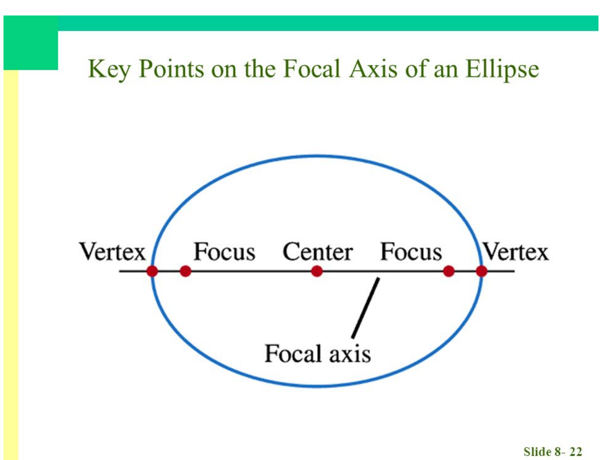
Planets follow elliptical orbits and telescopes are made using parabolic and hyperbolic lenses; these applications use **eccentricity** and the corresponding polar representations. Our goal will be modest: define eccentricity and write the conic relations in polar form.

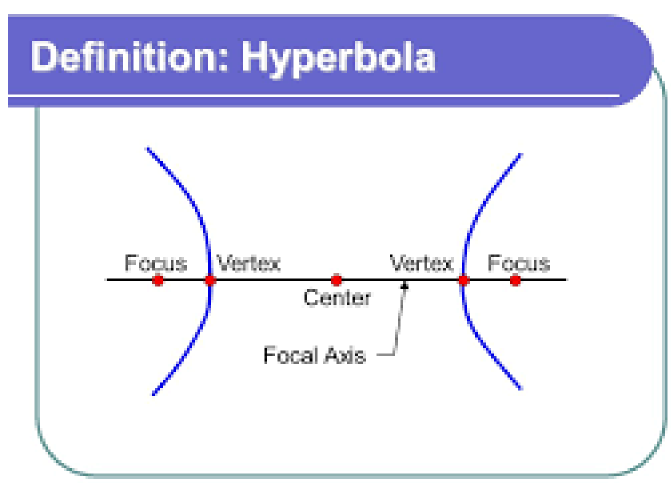




The eccentricity of an ellipse, hyperbola, or circle is

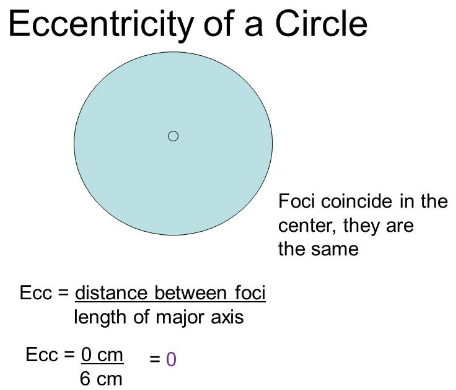
$$e = \frac{\text{distance between foci}}{\text{distance between vertices on focal axis}}.$$



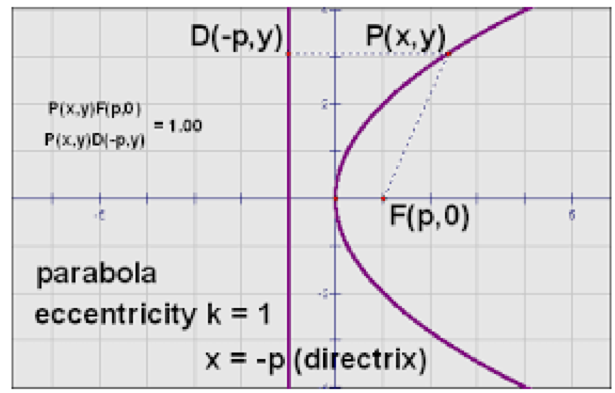


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ e < 1$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,\ e > 1$$



$x^2 + y^2 = R^2$, $e = 0$



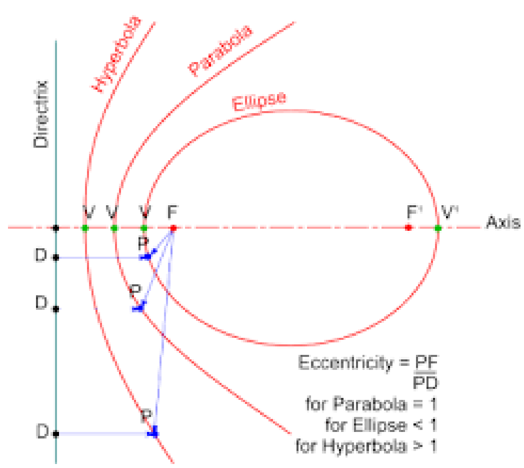
$x = ay^2$, $e = 1$

The eccentricity of a parabola is 1, but we need a different definition of eccentricity to see this. We need a **directrix**, one focus F and any point on the conic P:

$$e = \frac{\text{distance from } F \text{ to } P}{\text{distance from } F \text{ to the directrix}}.$$

This definition is equivalent to our original one for ellipses and hyperbolas.

Ellipses, hyperbolas, and parabolas have a directrix perpendicular to the focal axis. Circles don't have one in the real plane. For a conic with directrix, $e = \dfrac{PF}{PD}$ if P is any point on the conic and D is the closest point from P on the directrix.



This relation is equivalent to our familiar cartesian equations. If we place F at the origin **and the directrix to the right of** F, then with some work we can prove that the conic in polar coordinates is

$$r = \frac{ed}{1 + e\cos(\theta)} \qquad \text{(Standard Form)}$$

where $x = d > 0$ is the directrix. Notice the graph is always symmetric about the focal axis. If we want the focal axis to be vertical instead of horizontal, all we have to do is rotate C.C.W. by 90 degrees. Since $\cos(\theta - \pi/2) = \sin(\theta)$, we get

$$r = \frac{ed}{1 + e\sin(\theta)}$$

where $y = d > 0$ is the directrix. To get $y = -d < 0$ rotate 90 degrees C.W.; to get $x = -d < 0$ rotate 180 degrees.

What type of conic has $e = 0.5$ and directrix $x = -3$? What is the corresponding polar equation? How does the equation change if $x = 3$ or $y = -3$?

Find the directrix, eccentricity, and type of the conic that has the equation $r = \dfrac{12}{4 - 9\sin(\theta)}$. Rewrite in standard form first using a rotation.