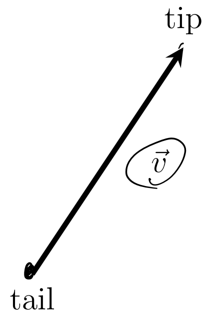
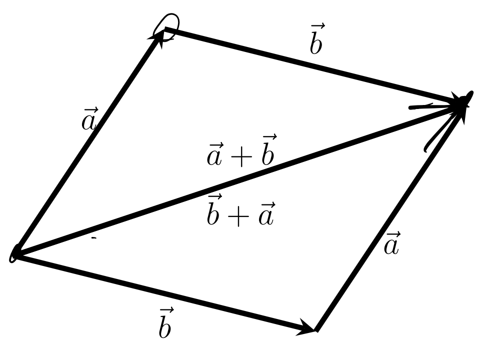
Vectors (HW #1)

A vector is an arrow with direction and length. Notation is $\vec{v}$ or a bold $\mathbf{v}$.



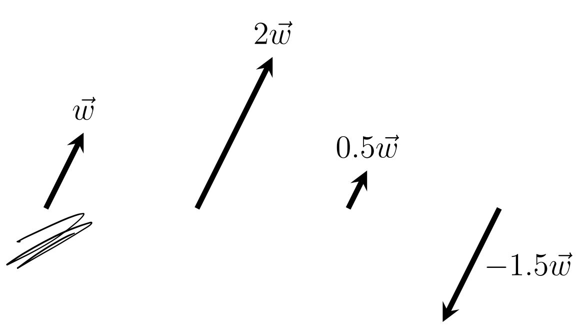
$\|\vec{v}\|$ = the **length** of the vector. **Magnitude** and **norm** are other words for "length."

Two vectors are added when the tail of the second is placed at the tip of the first with the sum equal to the vector with tail of the first and the tip of the second. The following picture shows that addition commutes: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.



Two vectors are **independent** if they are not parallel. The picture shows two independent vectors determine a parallelogram.

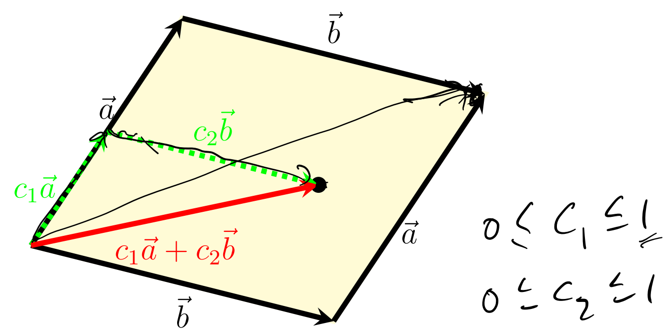
Scalar multiplication is a real number multiplied by a vector. It *rescales* the length of the vector. Negative numbers reverse direction, but otherwise the direction remains the same. All nonzero scalar multiples are **parallel** vectors.



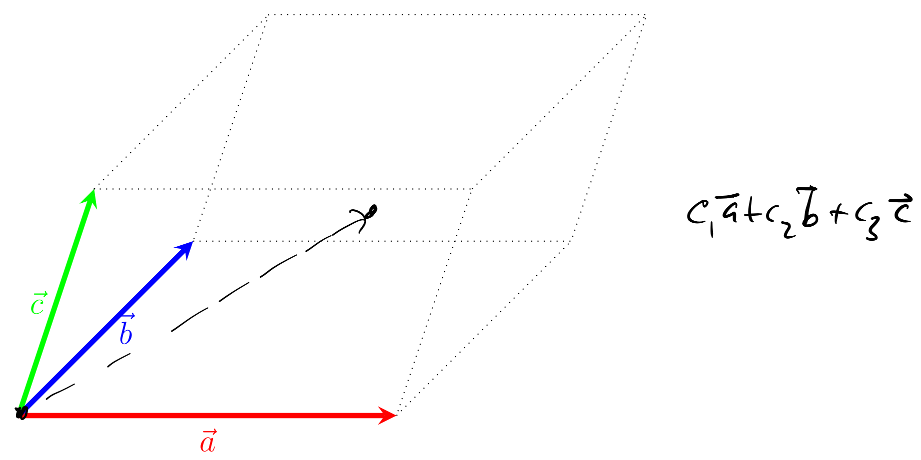
Here is some vocabulary we will use frequently in this class and in math 210.

$\vec{v}$ is a **unit vector** if $\|\vec{v}\| = 1$. Every nonzero vector $\vec{a}$ has a corresponding unit vector in the same direction: $\vec{e_a} = \dfrac{\vec{a}}{\|\vec{a}\|}$.

Every point in the parallelogram determined by $\vec{a}$ and $\vec{b}$ is the the tip of a sum of rescaled $\vec{a}$ and $\vec{b}$. We say that the parallelogram is **spanned** by $\vec{a}$ and $\vec{b}$ and that $c_1\vec{a} + c_2\vec{b}$ is a **linear combination** of $\vec{a}$ and $\vec{b}$.

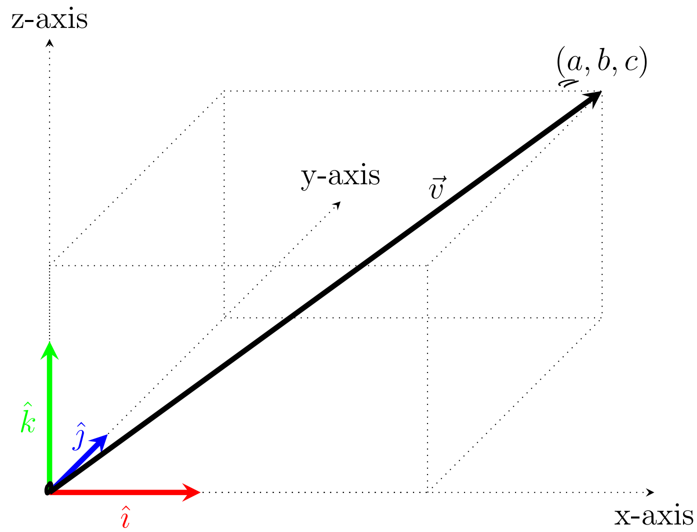


Vectors can be placed into any n-dimensional space, denoted $\mathbb{R}^n$, though we can only draw pictures of them in 1, 2, or 3-space. Here is a picture for $\mathbb{R}^3$:



We see that $\vec{a}$, $\vec{b}$, and $\vec{c}$ span a box (or parallelepiped) in 3-space, again because any point in the box can be determined by a linear combination of those three vectors.

In order to perform vector arithmetic easily we define vectors in terms of coordinates. To start this process, define unit vectors that are parallel to the coordinate axes. Conventional notation uses $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ to be unit vectors parallel to the x, y, and z-axes respectively. Now position any vector $\vec{v}$ with its tail at (0,0,0).

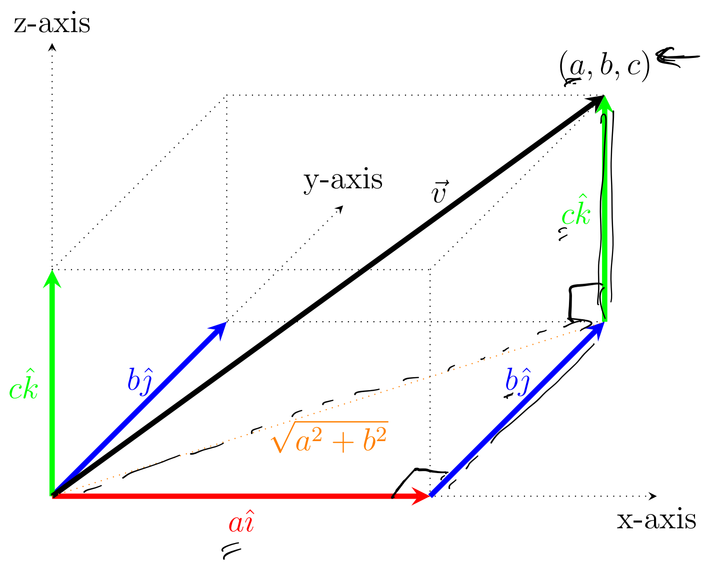


$\|\hat{\imath}\| = 1$

$\|\hat{\jmath}\| = 1$

$\|\hat{k}\| = 1$

Now we take a linear combination of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ using coefficients determined by the tip of $\vec{v}$ to decompose $\vec{v}$ into its **coordinate components**.



Then we have

$$\vec{v} = a\hat{\imath} + b\hat{\jmath} + c\hat{k} = \langle a, b, c\rangle \qquad \langle a, b, c\rangle$$

where $\langle a, b, c\rangle$ is called the component form for the vector. It is shorthand for $a\hat{\imath} + b\hat{\jmath} + c\hat{k}$.

Consequently, we have $\hat{\imath} = \langle 1, 0, 0\rangle$, $\hat{\jmath} = \langle 0, 1, 0\rangle$, and $\hat{k} = \langle 0, 0, 1\rangle$. Moreover, using Pythagorean's Theorem twice, we have

$$\|\vec{v}\| = \|\langle a, b, c\rangle\| = \sqrt{a^2+b^2+c^2}.$$

This arithmetic generalizes to all $\mathbb{R}^n$. In particular, in 2-space we have $\hat{\imath} = \langle 1, 0\rangle$, $\hat{\jmath} = \langle 0, 1\rangle$, and $\|\langle a, b\rangle\| = \sqrt{a^2+b^2}$.

$\lambda\langle a, b, c\rangle = \lambda(a\hat{\imath} + b\hat{\jmath} + c\hat{k}) = \lambda a\hat{\imath} + \lambda b\hat{\jmath} + \lambda c\hat{k} = \langle \lambda a, \lambda b, \lambda c\rangle$. Use this to prove that $\|\lambda\vec{v}\| = |\lambda|\|\vec{v}\|$ if λ is any constant.

$$\|\lambda\vec{v}\| = \|\lambda\langle a,b,c\rangle\| = \|\langle\lambda a, \lambda b, \lambda c\rangle\|$$

$$= \sqrt{(\lambda a)^2 + (\lambda b)^2 + (\lambda c)^2}$$

$$= \sqrt{\lambda^2(a^2+b^2+c^2)}$$

$$= |\lambda|\sqrt{a^2+b^2+c^2}$$

$$= |\lambda|\,\|\vec{v}\| \text{ if } \vec{v} = \langle a,b,c\rangle$$

Quickly simplify $\|\langle -25, -50, -50\rangle\|$.

$$= |-25|\,\|\langle 1,2,2\rangle\|$$

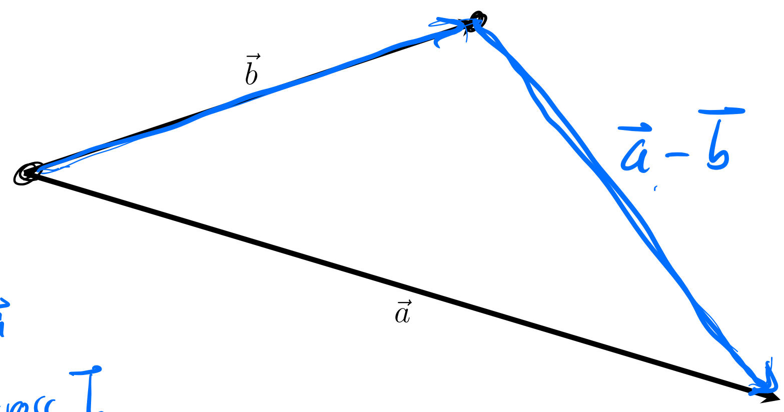
$$= 25\sqrt{1^2+2^2+2^2} = \boxed{75}$$

Simplify $5(\hat{\imath} - 2\hat{\jmath} + 2\hat{k}) - 2\langle 3, 4, -1\rangle$.

$$= \langle 5,-10,10\rangle + \langle -6,-8,+2\rangle$$

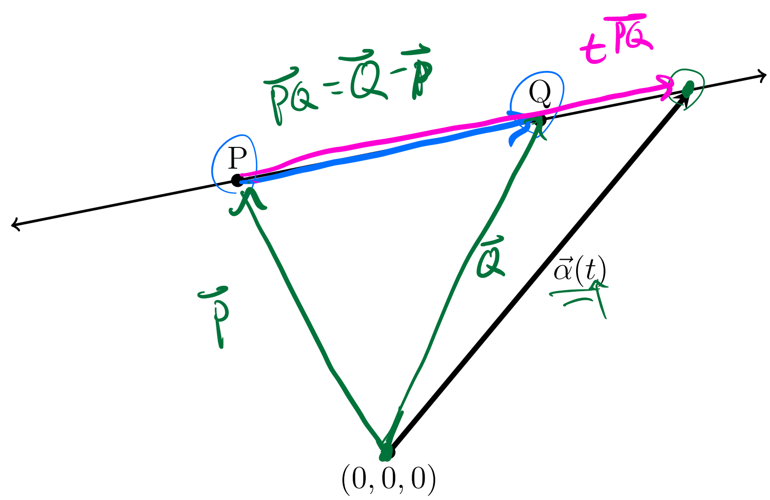
$$= \boxed{\langle -1,-18,12\rangle}$$

Draw $\vec{a} - \vec{b}$ on the picture below.



tail of $\vec{a}$
translated across $\vec{b}$.

Find a function $\vec{\alpha}(t)$ with outputs equal to vectors with tips on the line that contains the two points $P = (1, 2, 3)$ and $Q = (4, 8, 10)$. This function is called a **parameterization** for the line, or, if t represents time, it is also called a **position function** for the line. Define $\overrightarrow{PQ}$ to equal the vector with tail at P and tip at Q.



$$\vec{\alpha}(t) = \vec{P} + t\vec{PQ}$$

$$\vec{\alpha}(t) = \langle 1, 2, 3\rangle + t\left(\langle 4, 8, 10\rangle - \langle 1, 2, 3\rangle\right)$$

$$\boxed{\vec{\alpha}(t) = \langle \underbrace{1+3t}_{x(t)}, \underbrace{2+6t}_{y(t)}, \underbrace{3+7t}_{z(t)} \rangle}$$

What are the coordinate equations for the parameterization in the last example?

$$\boxed{\begin{aligned} x &= 1+3t \\ y &= 2+6t \\ z &= 3+7t \end{aligned}}$$

From the coordinate equations we can find the **symmetric form**. Solve for t in each coordinate equation and then set the three expressions equal.

$$\boxed{\begin{aligned} x &= 1+3t \\ y &= 2+6t \\ z &= 3+7t \end{aligned}} \qquad \begin{aligned} &\Rightarrow \frac{x-1}{3} = t \\ &\Rightarrow \frac{y-2}{6} = t \\ &\Rightarrow \frac{z-3}{7} = t \end{aligned}$$

$$\boxed{\frac{x-1}{3} = \frac{y-2}{6} = \frac{z-3}{7}}$$ is the symmetric form for the line.