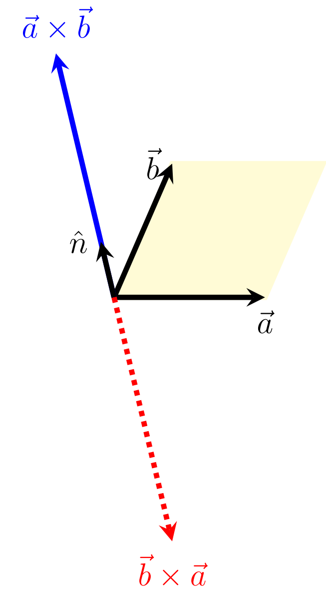
Cross Product (HW #1)

$$\vec{a} \times \vec{b} = \|\vec{a}\|\|\vec{b}\| \sin(\theta)\hat{n}$$

where $0 \leq \theta \leq \pi$ is the angle between $\vec{a}$ and $\vec{b}$, and $\hat{n}$ is a **unit** vector **perpendicular** to the plane spanned by $\vec{a}$ and $\vec{b}$ determined by the right-hand rule. The right-hand rule uses the right hand with fingers curling from $\vec{a}$ to $\vec{b}$ and thumb pointing in the direction of $\hat{n}$.



$\vec{a} \times \vec{b}$ is a **vector** while $\vec{a} \cdot \vec{b}$ is a scalar. The magnitude $\|\vec{a} \times \vec{b}\|$ equals the area of the parallelogram, and the direction is perpendicular to the plane spanned by $\vec{a}$ and $\vec{b}$.

Draw $\vec{b}_{a\perp}$ on the picture and notice how it is the height of the parallelogram. If $\vec{a}$ represents a wrench and $\vec{b}$ a force applied to the end of the wrench, then $\vec{b}_{a\perp}$ is the component of $\vec{b}$ that turns the wrench. The area of the parallelogram is a good measure of the turning force (or **torque**) since it increases with the length of the wrench and the length of $\vec{b}_{a\perp}$. The force would tighten a bolt with right-hand threads pointing in the direction of $\vec{a} \times \vec{b}$.

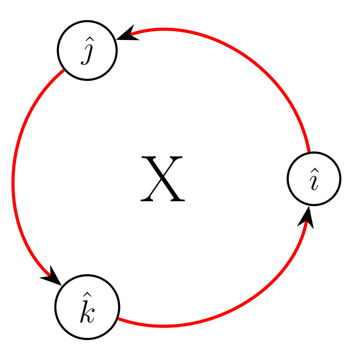
What factors of $\|\vec{a}\|\|\vec{b}\| \sin(\theta)\hat{n}$ multiply to equal the length of $\vec{b}_{a\perp}$?

Notice the right-hand rule guarantees the cross product is **not** commutative; rather, it is **anticommutative**:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

Find $\hat{\imath} \times \hat{\jmath}$, $\hat{\jmath} \times \hat{k}$, $\hat{k} \times \hat{\imath}$ using $\vec{a} \times \vec{b} = \|\vec{a}\|\|\vec{b}\| \sin(\theta)\hat{n}$.

Here is a nice summary of what we did; you need to add a negative to the answer if we multiply opposite the direction of the arrows.



Prove $(\lambda\vec{a}) \times \vec{b} = \lambda(\vec{a} \times \vec{b})$ if $\lambda \geq 0$. How does the proof change if $\lambda < 0$?

Find $\hat{\imath} \times \hat{\imath}$, $\hat{\jmath} \times \hat{\jmath}$, $\hat{k} \times \hat{k}$, and then write a generalization in terms of $\vec{a}$.

Fact: The cross product distributes over addition:

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}.$$

You can find the proof in **Second Year Calculus**, by David M. Bressoud. We assume it is true now and use it to find an easy way to calculate cross products if the vectors are given in component form.

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\vec{a} \times \vec{b} = (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k})$$

$$= (\qquad\qquad\qquad\qquad)\hat{\imath} + (\qquad\qquad\qquad\qquad)\hat{\jmath} + (\qquad\qquad\qquad\qquad)\hat{k}$$

$$= \begin{vmatrix} \qquad & \qquad \\ \qquad & \qquad \end{vmatrix}\hat{\imath} - \begin{vmatrix} \qquad & \qquad \\ \qquad & \qquad \end{vmatrix}\hat{\jmath} + \begin{vmatrix} \qquad & \qquad \\ \qquad & \qquad \end{vmatrix}\hat{k}$$

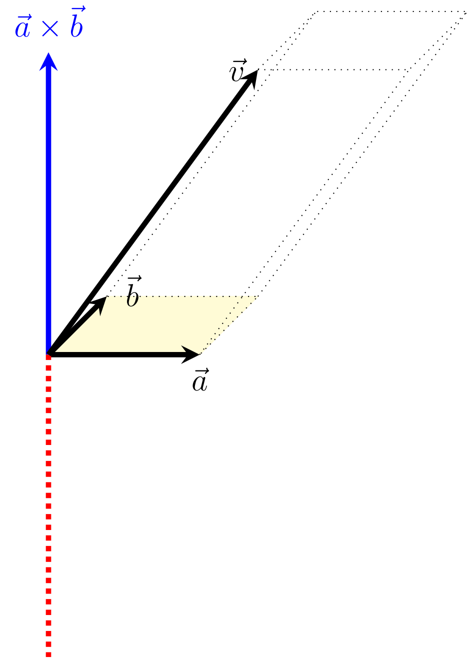
$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Find the area of the parallelogram spanned by $\vec{a} = \langle 1, 1, 2 \rangle$ and $\vec{b} = \langle 1, 3, 1 \rangle$. What is the area of the triangle spanned by $\vec{a}$ and $\vec{b}$?

Flux

The flux of a constant vector field $\vec{v}$ grams of water per second[1] through the parallelogram spanned by $\vec{a}$ and $\vec{b}$ (understood to be oriented from $\vec{a}$ to $\vec{b}$ by the right-hand rule) is the number of grams of water that flows through the parallelogram in one second. Flux can be negative or positive. The area of a box is base times height. Use the picture below to convince yourself that

$$\text{Flux} = \Phi = \vec{v} \cdot (\vec{a} \times \vec{b}).$$



Find the flux of a constant vector field $\vec{v} = \langle 4, 1, 2 \rangle$ grams of water per second through the parallelogram spanned by $\vec{a} = \langle 1, 1, 2 \rangle$ and $\vec{b} = \langle 1, 3, 1 \rangle$. What does the sign of the flux tell you?

What is the volume of the box spanned by $\vec{v} = \langle 4, 1, 2 \rangle$, $\vec{a} = \langle 1, 1, 2 \rangle$, and $\vec{b} = \langle 1, 3, 1 \rangle$?

[1] Flux can measure any type of flow. For instance, we might have coulombs of charge per second.

The order of operations for vector arithmetic remains the same as real numbers since arithmetic is done component-by-component. Sometimes one order is not defined and the other order is; in that case, it is understood that the defined order is used.

If $\vec{v} = \langle 4, 1, 2 \rangle$, $\vec{a} = \langle 1, 1, 2 \rangle$, and $\vec{b} = \langle 1, 3, 1 \rangle$, simplify

$$5\vec{a} \times \vec{v} \cdot (2\vec{a} - 4\vec{b}) + \|\vec{v} - 2\vec{a}\|$$

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