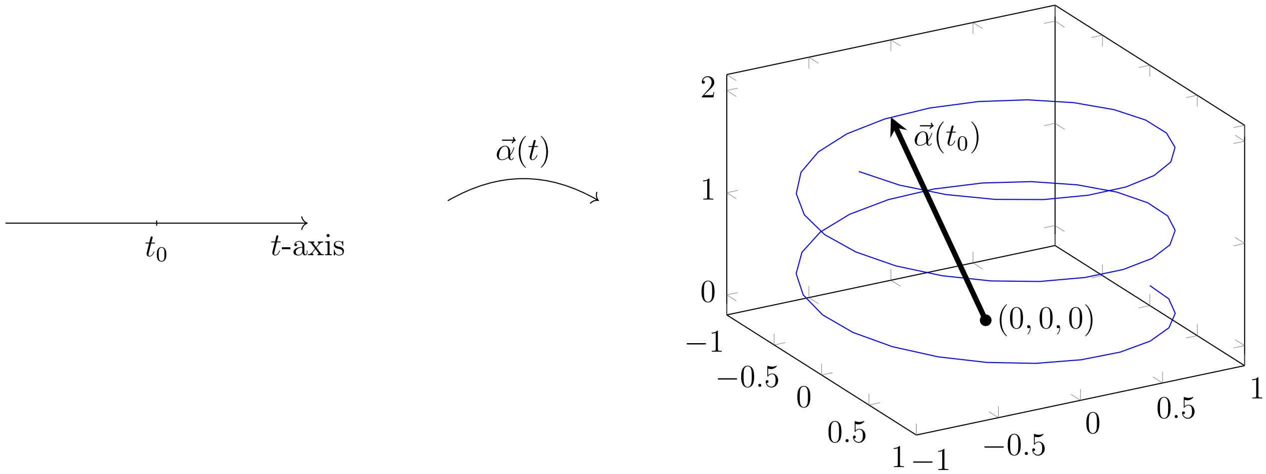
One Parameter Vector Functions (HW #2)



$\vec{\alpha}(t)$ is a vector function with one parameter if the output is a vector in $\mathbb{R}^n$. In the diagram above,

$$\vec{\alpha}(t) = \langle x(t), y(t), z(t) \rangle$$

maps one dimension into xyz-space. The curve in $\mathbb{R}^n$ is called the **trace** of $\vec{\alpha}(t)$, and the **coordinate equations** are $x = x(t)$, $y = y(t)$, and $z = z(t)$.

Such functions have lots of uses, but we can think of them intuitively as position functions where $\vec{\alpha}(t_0)$ is a vector that points to a position on the curve at time t_0. The orientation of the t-axis determines the orientation of the trace of $\vec{\alpha}(t)$ in $\mathbb{R}^n$.

When I ask you to **parameterize a curve** I want you to construct a vector function, or give coordinate equations for one, that has a trace equal to the curve.

Parameterize the line that contains the points $P = (1, 2, 1)$ and $Q = (4, 2, -2)$. What are the coordinate equations? What is the orientation? Give another parameterization that reverses the orientation. How do we parameterize the line segment from P to Q?

Parameterize the circle of radius 2 that starts and ends at $(2, 0)$. Change the parmeterization so it starts at $(0, 2)$. How do I change the orientation? Find a parameterization for a particle that rotates around this circle in 10 seconds.

Parameterize the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

Find a parameterization for the graph of $y = \cos(x)$. Generalize to any function in the form of $y = f(x)$ or $x = g(y)$.

The trace of $\vec{\alpha}(t) = \langle 1, \tan(t), -\sec(t) \rangle$ for $\frac{-\pi}{2} < t < \frac{\pi}{2}$ is the intersection of two familiar surfaces. What are they? Draw one of the surfaces and then sketch the trace of $\vec{\alpha}(t)$ on it. Indicate the orientation of the trace. See matlab notes for homework 2.

Calculus on vector functions is performed component-by-component

Let $\vec{\alpha}(t) = \langle x(t), y(t), z(t)\rangle$. We define

$$\lim_{t\to a}\langle x(t), y(t), z(t)\rangle = \langle \lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t)\rangle.$$

Then vector arithmetic implies

$$\vec{\alpha}'(t) = \frac{d\vec{\alpha}(t)}{dt} = \langle x'(t), y'(t), z'(t)\rangle$$

and

$$\int \vec{\alpha}(t)\,dt = \left\langle \int x(t)\,dt, \int y(t)\,dt, \int z(t)\,dt \right\rangle.$$

Find the derivative and indefinite integral of $\vec{\alpha}(t) = \langle \cos(t), \sin(t), t\rangle$.

Product rules

If $k(t)$ is a real-valued function and $\vec{\alpha}(t)$ and $\vec{p}(t)$ are vector valued functions, then:

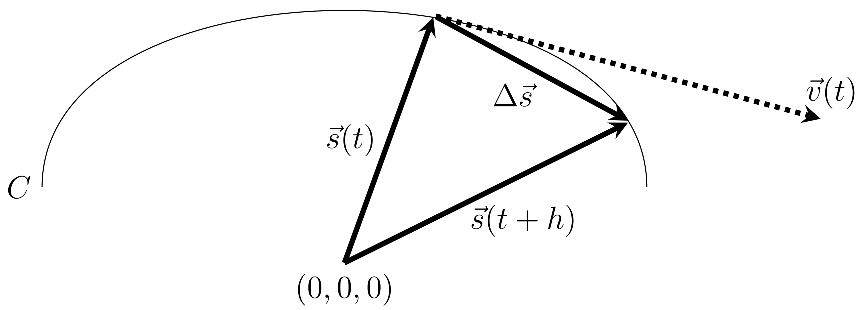
$$(k(t)\vec{\alpha}(t))' = k'(t)\vec{\alpha}(t) + k(t)\vec{\alpha}\,'(t)$$

$$(\vec{p}(t) \cdot \vec{\alpha}(t))' = \vec{p}\,'(t) \cdot \vec{\alpha}(t) + \vec{p}(t) \cdot \vec{\alpha}\,'(t)$$

$$(\vec{p}(t) \times \vec{\alpha}(t))' = \vec{p}\,'(t) \times \vec{\alpha}(t) + \vec{p}(t) \times \vec{\alpha}\,'(t)$$

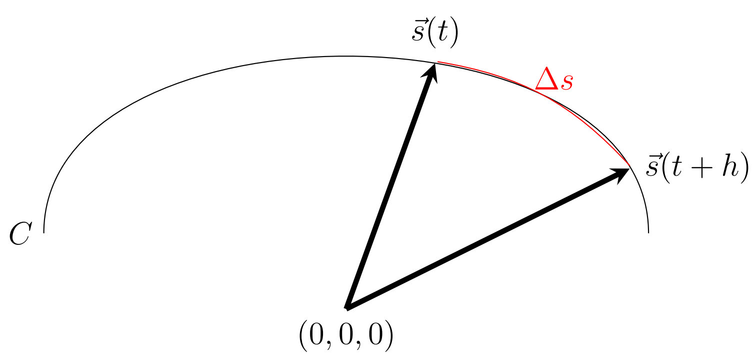
Find $(k\vec{p})'(0)$, $(\vec{p} \cdot \vec{\alpha})'(0)$, and $(\vec{p} \times \vec{\alpha})'(0)$ if $k(t) = t^2 - t + 1$, $\vec{\alpha}(t) = \langle \cos(t), \sin(t), t \rangle$ and $\vec{p}(t) = \langle \sin(t), -\cos(t), t \rangle$.

If $\vec{s}(t)$ represents the **position** of a particle at time t, then $\vec{v}(t) = \vec{s}\,'(t) = \dfrac{d\vec{s}}{dt}$ is the **velocity** of the particle at time t. The picture below shows $\Delta\vec{s} = \vec{s}(t+h) - \vec{s}(t)$ has direction becoming closer to the tangent line and pointing in the direction in which the particle is moving as $h \to 0^+$. Since $\vec{v} = \lim_{h\to 0} \dfrac{\Delta\vec{s}}{h}$, $\vec{v}(t)$ is tangent to the trace of $\vec{s}(t)$ and pointing in the direction of orientation.



We assume the particle does not stop, that is, $\vec{v} \neq \vec{0}$. Then the arc length is denoted s and the speed by $v(t) = \|\vec{v}(t)\| = \dfrac{ds}{dt}$.[1] If time changes a small amount, then $\Delta s \approx v\Delta t$. The arc length of the curve C parameterized by $\vec{s}(t)$ for $t_0 < t < t_f$ is approximately $\sum \Delta s$ so passing to a limit and then substituting we obtain

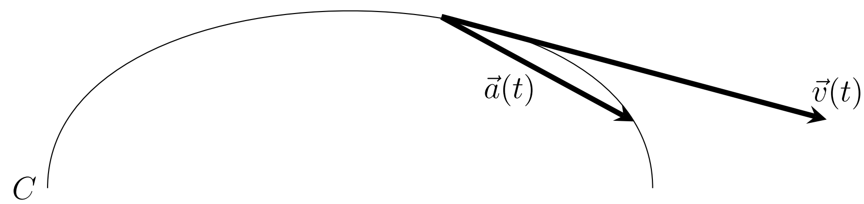
$$\text{arc length} = s = \int_C ds = \int_{t_0}^{t_f} v\,dt.$$



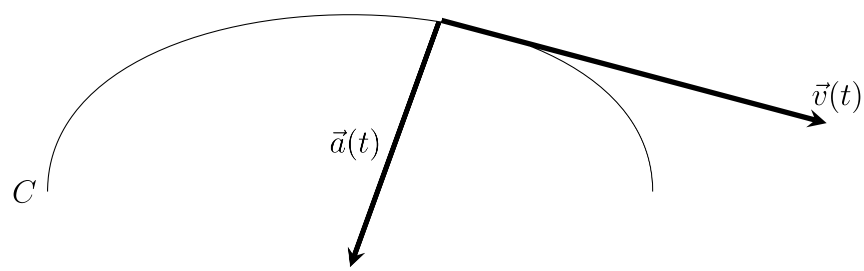
[1] s is not $\|\vec{s}\|$, but v **does** equal $\|\vec{v}\|$.

Find the arc length of $\vec{s}(t) = \langle \cos(t), \sin(t), \frac{2t^{3/2}}{3} \rangle$ for $0 \le t \le 8$.

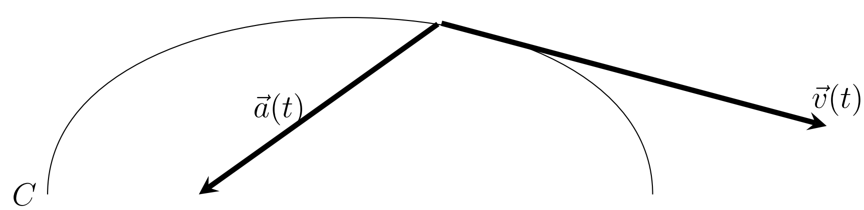
The acceleration of the particle is $\vec{s}''(t) = \vec{a}(t)$. It points in the same direction as the force on the particle that drags it away from straight line motion, so $\vec{a}(t)$ must point from the particle into the cave of the curve. There are three cases, one for each different type of projection.



Speeding-up: $0 \le \theta \le \pi/2$



No change in speed: $\theta = \pi/2$



Slowing down: $\pi/2 \le \theta \le \pi$

For an instant, the path of the particle stays within the plane determined by its position (a point) and the two vectors $\vec{v}$ and $\vec{a}$. This is the **osculating (or kissing) plane**.

$\hat{T}(t) = \dfrac{\vec{v}}{v}$ is the **unit tangent vector** to the curve at $\vec{s}(t)$, and $\hat{N}(t) = \dfrac{\vec{a}_{v\perp}}{\|\vec{a}_{v\perp}\|}$ is the **unit normal vector**. These two vectors act locally like $\hat{\imath}$ and $\hat{\jmath}$ - as a coordinate basis. Mark $\hat{T}(t)$ and $\hat{N}(t)$ at different points on the above curves. For one of those points sketch a circle through it so that its tangent line at that point is parallel to the velocity vector. This is the **osculating circle**, the circle that best fits the curve at that point and lies in the osculating plane. Its radius R is inversely proportional to how curved the trace is; we define the **curvature**, $\kappa(t)$, at every point on the trace pointed to by $\vec{s}(t)$ to be

$$\kappa = \frac{1}{R}$$

where R is the radius of the osculating circle at that point.

Find the osculating plane, $\hat{T}(t)$, and $\hat{N}(t)$ for $\vec{s}(t) = \langle \cos(t), \sin(t), t \rangle$ at time $t = 2\pi$.

How to calculate curvature?

We can prove, given time, that the following are alternative definitions of curvature, equal to $\kappa = \dfrac{1}{R}$. Notice the units of $1/R$ are 1/meters and so are the units of the alternate definitions below. Both can be intuitively associated with the curve of the trace. How?

1) $$\kappa = \frac{\|\vec{a} \times \vec{v}\|}{v^3}$$

2) $$\kappa = \left\|\frac{d\hat{T}}{ds}\right\| = \left\|\frac{d\hat{T}}{dt}\right\| \left\|\frac{dt}{ds}\right\| = \frac{\|\frac{d\hat{T}}{dt}\|}{v}$$

I usually use the first formula when calculating curvature. The second is more useful when discussing the theory of curvature.

Find κ and the radius of the osculating circle at $t = 1$ for the trace of $\vec{s}(t) = \langle t^2, t, t^2 \rangle$.