Line Integrals (HW #3)

A **line integral** is integrated over a curve C parameterized by $\vec{s}(t)$ for $a \leq t \leq b$.

Scalar Line Integrals We use the notation $\int_C f(s)ds$ where s is arc length of C. To evaluate the integral we remember $\frac{ds}{dt} = \|\vec{s}'(t)\|$ and substitute $s = s(t)$ and $ds = \|\vec{s}'(t)\|dt$. The substitution is sometimes called the "pullback" from n-space to t-space.

Find the mass of a wire C with linear density $\delta(x, y, z) = 3z^2$ grams per centimeter if $C : \vec{s}(t) = \langle \sin(t), 3t, \cos(t) \rangle$ and $0 \leq t \leq \pi/2$.

Let C be the curve $y = \sqrt{x}$ from $x = 2$ to $x = 6$. Evaluate $\int_C y\, ds$.

Vector line integrals calculate work, not mass, and so need to sum up dot products. In this case we have $\int_C \vec{F} \cdot d\vec{s}$. Recall the curve C is parameterized by $\vec{s}(t)$ for $a \le t \le b$ so that $d\vec{s} = \frac{d\vec{s}}{dt} dt$. Substitution then gives

$$\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F} \cdot \vec{s}\,'(t)\, dt.$$

Find the work done by $\vec{F}(x, y) = \langle x \sin(y), \cos(x) \rangle$ on a particle that moves from $(0, 0)$ to (π, π^2) on the curve $y = x^2$.

Will the work change if we change the path but keep the endpoints the same? Calculate the work for a new path C_1 parameterized by $\vec{s}(t) = \langle t, \pi t \rangle$. Vector fields that depend only on the endpoints, not the path, to calculate work are **gradient vector fields**. Is $\vec{F}(x, y) = \langle x \sin(y), \cos(x) \rangle$ a gradient field?

Differential Notation

$d\vec{s} = \vec{s}\,'(t)\,dt = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$ so that

$$d\vec{s} = \langle dx, dy, dz\rangle.$$

If $\vec{F} = \langle P, Q, R\rangle$ then

$$\vec{F} \cdot d\vec{s} = Pdx + Qdy + Rdz$$

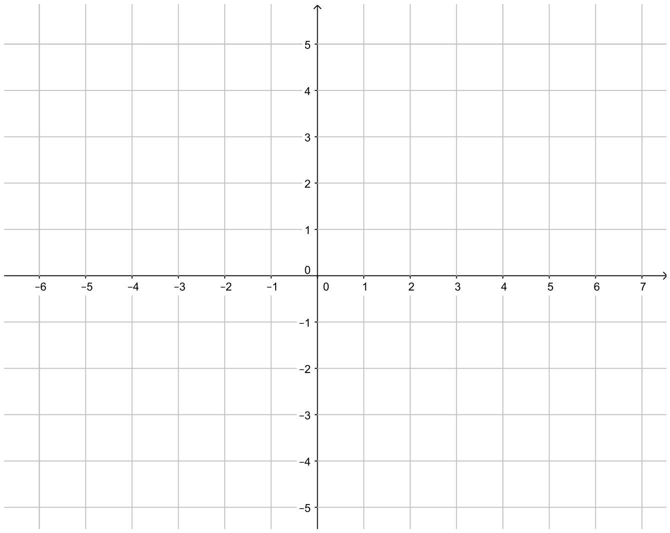
and

$$\int_C \vec{F} \cdot d\vec{s} = \int_C Pdx + Qdy + Rdz.$$

This is the differential form for a vector line integral.

Find $I = \displaystyle\int_C yzdx + xdy + ydz$ if C is the circle oriented CCW with respect to the z-axis found by intersecting the cylinder $r = 1$ with the hemisphere $z = \sqrt{4 - x^2 - y^2}$.

Sketch $\vec{F}(x,y) = \dfrac{\langle y, -x \rangle}{\sqrt{x^2+y^2}}$.



Use the sketch to determine if $\displaystyle\int_C \vec{F}(x,y) \cdot d\vec{s}$ for $C : \vec{s}(t) = \langle 4\cos(t), 4\sin(t) \rangle$ and $0 \le t \le 2\pi$ is positive, negative or zero. Then verify your estimate by calculating the integral.