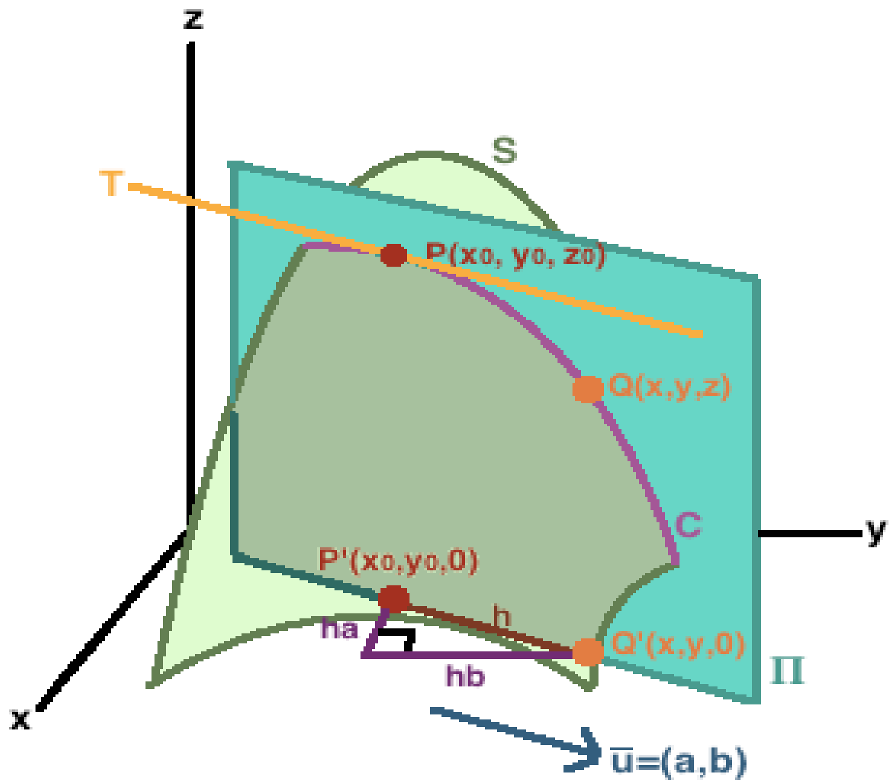
Directional Derivatives and the Gradient (HW #4)

How does f(x,y) change in the direction of unit vector $\hat{u}=\langle a,b\rangle$, $\|\hat{u}\|=1$?



It should be the slope of the line that lies on the tangent plane over $\hat{u}$. But that is the rise divided by a run of any distance along $\hat{u}$ in the xy-plane. Let's just run the length of $\hat{u}$. Then the **directional derivative in the direction** $\hat{u}=\langle a,b\rangle$ **at the point** (x_0,y_0) is

$$\left.\frac{\partial f}{\partial \hat{u}}\right|_{(x_0,y_0)} = D_{\hat{u}} f(x_0,y_0) = \frac{f_x(x_0,y_0)dx + f_y(x_0,y_0)dy}{1} = f_x(x_0,y_0)a + f_y(x_0,y_0)b = \nabla f(x_0,y_0)\bullet\hat{u}$$

So we can find the rate of change in any direction using the nifty formula

$$D_{\hat{u}} f = \nabla f \cdot \hat{u}$$

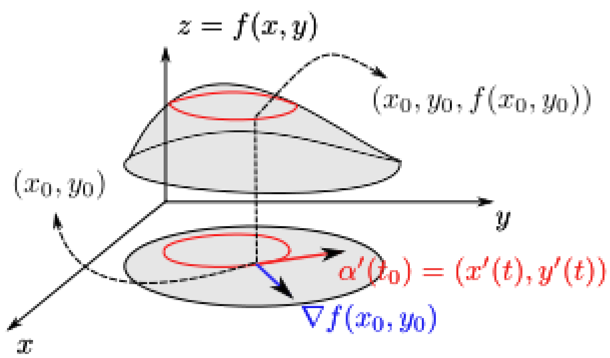
where $\hat{u}$ is a unit vector.

Let $g(x,y) = 2x + y + \ln(xy)$. Find $D_{\vec{u}}g(1,1)$ if $\vec{u}$ is in the same direction as $\langle -1,1 \rangle$.

How does the directional derivative change if the direction moves from (1, 1) to (-1, 1)?

Three properties of gradients related to the properties of dot products.

Property 1: $\nabla f(x_0, y_0)$ **is perpendicular to the level curve of** $f(x, y)$ **that passes through** $(x_0, y_0, 0)$**.** The function does not change height if the inputs stay (instantaneously) on the same level curve, so the rate of change is 0 if $\hat{u}$ is parallel to $\vec{\alpha}'(t_0)$ in the picture below. But a zero dot product indicates the vectors are perpendicular, and so $\nabla f(x_0, y_0)$ is perpendicular to the tangent line of the level curve.



The same reasoning tells us that $\nabla g(x_0, y_0, z_0)$ **is perpendicular to the level** ***surface*** **of** $g(x, y, z)$ **that passes through** $(x_0, y_0, z_0, 0)$**.**

Find the equation of the tangent line to the curve $x^2 + y^3 = 9$ at the point $(1, 2)$.

Find the equation of the tangent plane to the surface $3+x+y+z=\cos(x)+\cos(y)+\cos(z)$ at the point $(0,0,0)$.

Property 2: $\nabla f(x_0,y_0)$ **points from** (x_0,y_0) **in the direction of maximal rate of increase per input of** $f(x,y)$**.**

$$D_{\hat{u}}f=\nabla f\cdot\hat{u}=\|\nabla f\|\|\hat{u}\|\cos\theta=\|\nabla f\|\cos\theta$$

which is largest when $\theta=0$. Hence $\hat{u}$ has the same direction as the gradient and the magnitude of $\nabla f(P)$ is equal to that maximal rate.

Property 3: $-\nabla f(x_0,y_0)$ **points from** (x_0,y_0) **in the direction of minimal rate of increase per input of** $f(x,y)$**.**

$$D_{\hat{u}}f=\nabla f\cdot\hat{u}=\|\nabla f\|\|\hat{u}\|\cos\theta=\|\nabla f\|\cos\theta$$

which is smallest when $\theta=\pi$. Hence $\hat{u}$ has the opposite direction as the gradient and $-\|\nabla f\|$ is equal to that minimal rate.

Sketch an estimate of the gradient at the points $(2.5, 0.5)$ and $(0, -1)$ on the following contour plot.