## The Chain Rule (HW #4)

Matrices and Linear Algebra are needed if we wish to define the most general chain rule rigorously. Instead we describe a few special cases. The big idea implies the general rule, though: **the change in $f(x_1, x_2, \ldots, x_n)$ with respect to a new variable from composition is a sum of terms**, **one for every input variable**.

The differential gives formulas easily remembered if we treat differentials like numbers.

For instance, if z = f(x, y), x = x(t), and y = y(t), then without rigor we have

$$\frac{dz}{dt} = \frac{z_x dx + z_y dy}{dt} = z_x \cdot \frac{dx}{dt} + z_y \cdot \frac{dy}{dt}, \text{ so}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Find $\frac{df}{dt}$ at $t = \pi/3$ if f(x, y) = xy, x = sin(t), $y = y(t)$, $y(\pi/3) = 2$ and $y'(\pi/3) = 5$.

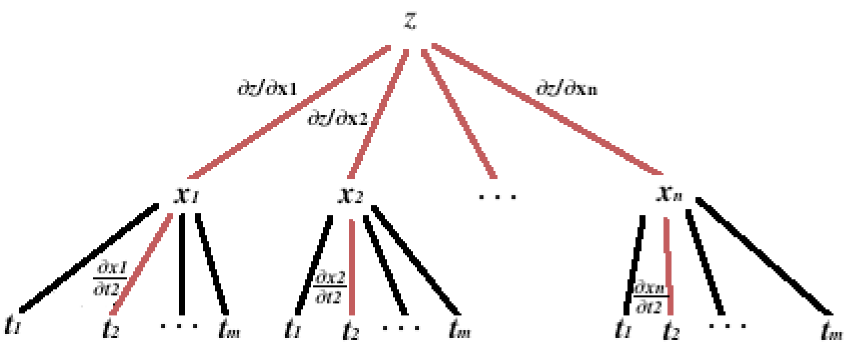
Similarly, if z = f(x, y), x = x(u,v), and y = y(u,v), then we have

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

and

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

We can use a diagram, called the dependence tree, to show the terms in a derivative of a composition:




***A tree diagram can be useful in memorizing the chain rule. Place the dependent variable on the top, the intermediate variables in the middle, and the independent variables (the parameters) at the bottom.***

***For example, to compute $\partial z/\partial t_2$, take the products from z to $t_2$ and then sum them.***

$$\partial z/\partial t_2 = [\partial z/\partial x_1][\partial x_1/\partial t_2] + [\partial z/\partial x_2][\partial x_2/\partial t_2] + \ldots + [\partial z/\partial x_n][\partial x_n/\partial t_2]$$

Find $\frac{\partial u}{\partial x}$ at $x = 1$, $y = 2$, and $t = \pi$ if $u = \sqrt{r^2 + s^2}$, $r = y + x\cos(t)$, and $s = 3x + y\sin(t)$.

$m(x,y)=4F(xy^2,1+xy)-3G(e^{xy})$. Find $m_x(0,2)$ and $m_y(0,2)$ if $F_x(0,1)=10$, $F_y(0,1)=5$, and $G'(1)=2$. You must assume that $F=F(x,y)$ has been composed with other functions so that $F_x(0,1)=10$ means the partial derivative of $F$ with respect to the first component.