Gradient Fields (HW #5)

If $\vec{F} = \nabla\phi$, then $\vec{F}$ is a **gradient field** with **potential** ϕ.

So if $\vec{F}(x, y, z) = \langle P, Q, R\rangle$ has a potential ϕ, then $P = \phi_x$, $Q = \phi_y$, and $R = \phi_z$. Similarly, if $\vec{F}(x, y) = \langle P, Q\rangle$ has a potential ϕ, then $P = \phi_x$ and $Q = \phi_y$. A similar statement is true for $\vec{F}(x_1, x_2, \ldots, x_n)$.

Find the gradient field $\vec{F}$ with potential $\phi = xy^2 + z^4$.

Finding a potential for a gradient field is a bit harder.

Method 1: Guess and Check.

Integrate each component, match the results, guess, and check.

$\vec{F}(x, y) = \langle 2xy^2 - 3x^2, 2x^2y + \cos(y)\rangle$ is a gradient field. Find a potential for it.

Method 2: Integrate/Differentiate.

$\vec{F}(x, y) = \langle y\sin(x^2) + 2x^2 y\cos(x^2) + 1, x\sin(x^2)\rangle$ is a gradient field. Find a potential for it.

Fundamental Theorem of Line Integrals (FTCLI)

If C is a curve parameterized by $\vec{\alpha}(t)$ for $a \le t \le b$, and if $\alpha(t)$ is the point at the tip of $\vec{\alpha}(t)$, then

$$\int_C \nabla f \cdot d\vec{s} = f(\alpha(b)) - f(\alpha(a)).$$

The proof uses the chain rule. I will prove it for two inputs. Recall that $\vec{\alpha}(t) = \langle x(t), y(t)\rangle$.

$$\int_C \nabla f \cdot d\vec{s} = \int_a^b \nabla f \cdot \vec{\alpha}\,'(t)\,dt = \int_a^b \langle f_x, f_y\rangle \cdot \langle x'(t), y'(t)\rangle dt$$

$$= \int_a^b f_x x'(t) + f_y y'(t) dt = \int_a^b \frac{d\,f(\alpha(t))}{dt} dt = f(\alpha(b)) - f(\alpha(a)).$$

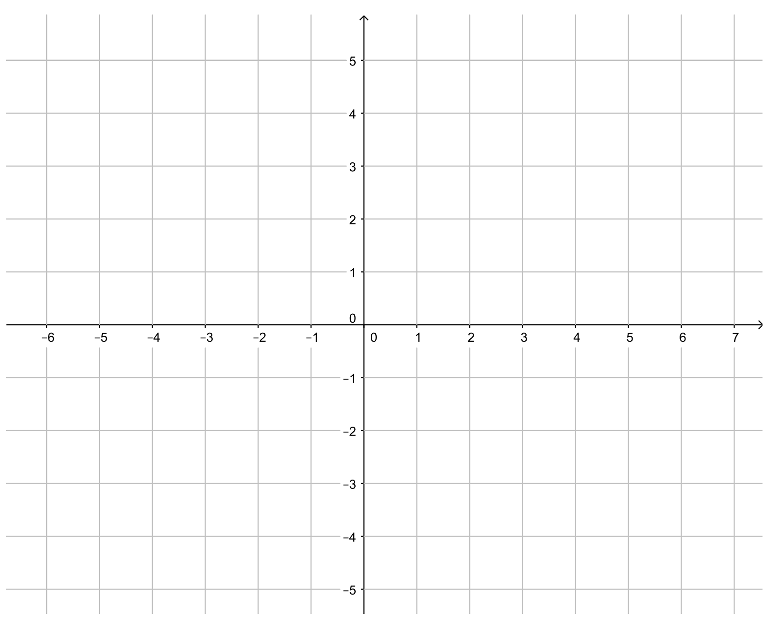
Calculate $I = \int_C \langle y, x \rangle \cdot d\vec{s}$ if C is parameterized by $\vec{\alpha}(t) = \langle \cos(\pi t), t^2 \rangle$ for $0 \le t \le 2$.

How does your answer change if C is **any** path from $(1, 0)$ to $(1, 4)$?

Consequently every gradient field is **path independent**. What would the line integral equal if C were any path from $(1, 4)$ to $(1, 4)$?

We say every gradient field is **conservative** since the work done by it on a particle about any closed curve is zero and hence the change in energy is zero.

Sketch the graph of the vector field $\vec{F}(x, y) = \dfrac{-y\hat{\imath} + x\hat{\jmath}}{x^2 + y^2}$. Is $\vec{F}$ a gradient field? Pay special attention to the vectors on the unit circle. Find the line integral about the unit circle to verify your answer.



$\vec{F}(x,y,z) = \langle y\cos(z) + yz\cos(x), x\cos(z) + z\sin(x), y\sin(x) - xy\sin(z)\rangle$ is a gradient field. Find a potential using the FTCLI and differential form. This is a third method for finding a potential.

Start by choosing any point in the domain of $\vec{F}$. I will choose $(0,0,0)$ and then set $\phi(0,0,0) = 0$. Then path independence allows us to choose any path in the domain of $\vec{F}$ from $(0,0,0)$ to (a,b,c): we choose a piecewise-defined path from $(0,0,0)$ to $(a,0,0)$ to $(a,b,0)$ to (a,b,c). Then FTCLI implies

$$\phi(a,b,c) = \phi(a,b,c) - \phi(0,0,0) = \int_0^a 0\,dx + \int_0^b a\,dy + \int_0^c b\sin(a) - ab\sin(z)\,dz.$$

Finish the problem. Notice this method only works if you pay special attention to the domain of the vector field.

So far I have told you when a vector field is a gradient field or not. How can we determine this without finding a potential function? We assume the mixed partials of the potential are equal; then

$$\nabla \times \nabla\phi = \langle 0, 0, 0\rangle.$$

Verify this.

Curl Test if the domain of $\vec{F}$ is open, path connected, and simply connected, then $\vec{F}$ is a gradient field if and only if $\nabla \times \vec{F} = \vec{0}$.

The domain of $\vec{F}(x, y, z) = \langle y\cos(z) + yz\cos(x), x\cos(z) + z\sin(x), y\sin(x) - xy\sin(z)\rangle$ is $\mathbb{R}^3$ which is open, path connected, and simply connected. Use the curl test to show $\vec{F}$ is a gradient field.

A domain is **open** if it contains a disc about every point.

A domain is **path connected** if every pair of points has a path in the domain connecting them.

A domain is **simply connected** if every closed curve in the domain can be continuously squeezed to a point without leaving the domain.

We have already seen that $\vec{F}(x,y) = \dfrac{-y\hat{\imath} + x\hat{\jmath}}{x^2+y^2}$ cannot be a gradient field since it is not conservative. However, the curl of $\vec{F}$ is $\vec{0}$. Verify this.

The domain of $\vec{F}(x,y) = \dfrac{-y\hat{\imath} + x\hat{\jmath}}{x^2+y^2}$ must not meet the demands of the curl test's hypotheses. Which one is violated?