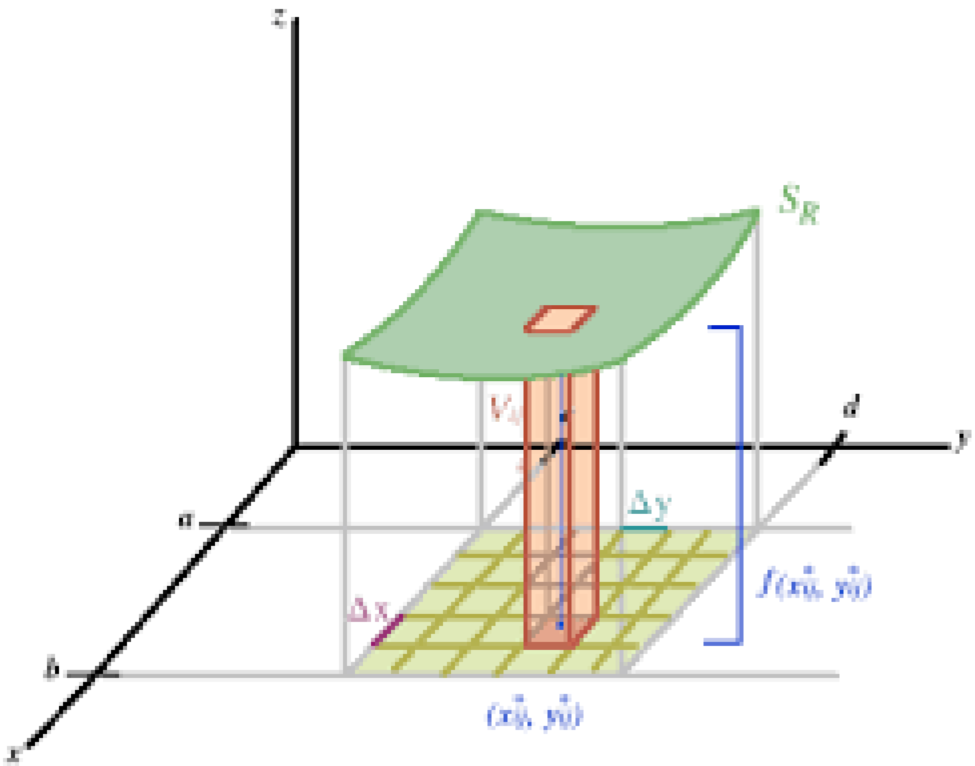
# Multivariable Integrals (HW #5)

How can we find the net volume underneath the graph of $z = f(x,y)$ over a region of integration $R$? Similar to integrals from single variable calculus we partition $R$ into small rectangles $A_i$ with sides parallel to the $x$ and $y$ - axes of length $\Delta x$ and $\Delta y$. Then we sum up the net volumes of all the boxes determined by the rectangles. This is an estimate; as the maximum diameter of the $A_i's \to 0$, the estimate becomes exact. $P$ will stand for "partition" and $\|P\|$ will equal the maximum diameter of the $A_i's$ in that partition.



We define

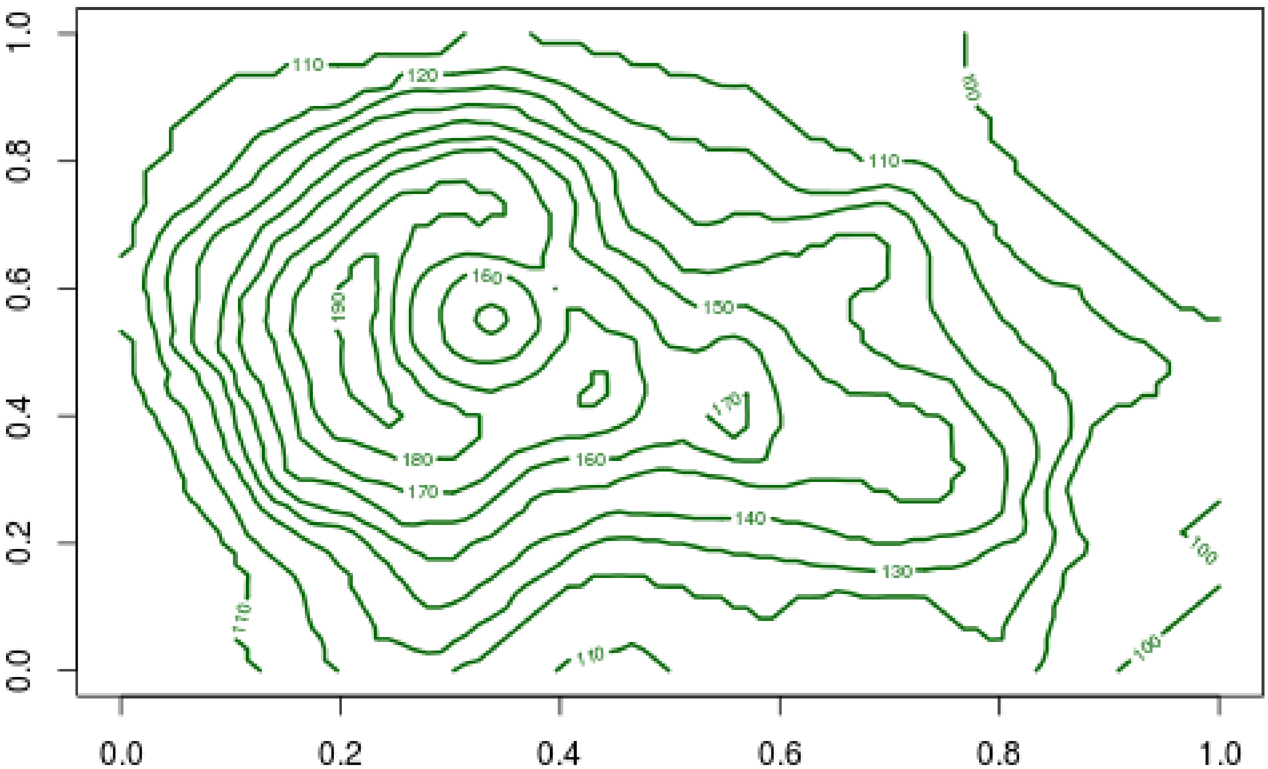
$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dxdy = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i,y_i)\,\Delta x \Delta y.$$

The right side of the definition is a Riemann sum, and the points $(x_i, y_i)$ are chosen arbitrarily from the rectangle $A_i$.

You can think of the double integral intuitively as a net volume or as a net mass where positive mass represents matter and negative mass represents antimatter.

We can use the Riemann sum to estimate a double integral.

Choose $(x_i, y_i)$ on a labeled level curve while estimating $\iint_R f(x,y)\,dA$ if $f(x,y)$ has the following contour plot and $R = [0.2, 0.6] \times [0.4, 0.8]$ with $\Delta x = \Delta y = 0.2$.



Double Integrals can be calculated by integrating a single variable integral twice. We can choose to integrate with respect to $x$ first or with respect to $y$ first. Which variable are we integrating over first in the following picture?

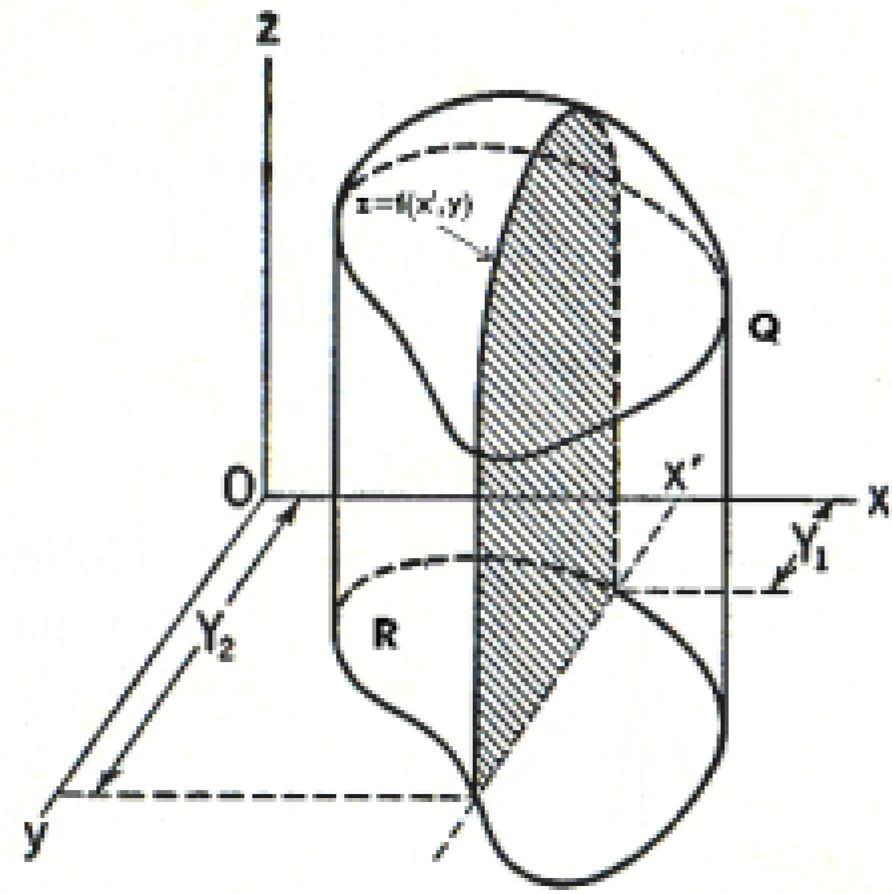



Fig. 4

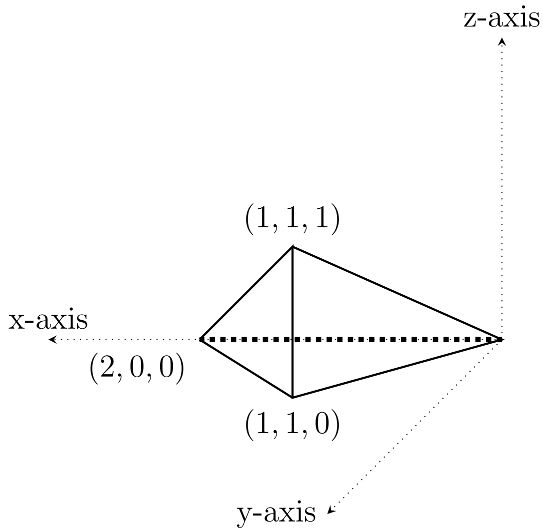
Instead of drawing this picture for each integral, draw the region $R$ in the $xy$-plane and a generic slice to find maximum and minimum coordinates for each integral.

Label maximum and minimum coordinates on the picture above and set up the corresponding double integral.

Calculate $I = \iint\limits_R x + yx^2 dA$ if $R$ is bounded by $y = 2x$, $y = 2$, and $x = 0$. If time allows, calculate twice using different orders.

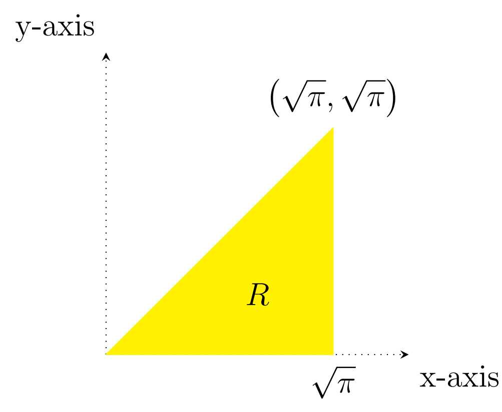
$I = \iint\limits_R dA =$ area of $R$. Verify this for the above example. The volume under the graph of the function $f(x, y) = 1$ is numerically equal to the area of the region of integration.

Use a double integral to find the volume of the solid bounded by $z = y$, $y = x$, $x + y = 2$, and $z = 0$. While attempting to draw this pyrymid you should notice the vertices are $(1, 1, 1)$, $(2, 0, 0)$, $(0, 0, 0)$, and $(1, 1, 0)$. Check your answer by using a well known formula derived using integrals: volume of pyrymid $= \frac{1}{3}$Base $\cdot$ Height.



Sketch the region of integration and then rewrite $I = \int_0^2 \int_{-y}^{y^2} f(x,y)\,dx\,dy$ by switching the order of integration.

Calculate $\iint_R \sin(x^2)\,dx\,dy$ if $R$ is the shaded region below.



## Triple Integrals in Rectangular Coordinates

The definition of triple integrals for functions of three variables is similar to double integrals except we have a partition of cubes and sum up 4-dimensional "content" instead of volumes. Intuitively, it might be better to think of the integrand as a density and the triple integral as a net mass of the solid E.

$$\iiint_E f(x,y,z)\,dV = \iiint_E f(x,y,z)\,dx\,dy\,dz = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(x_i,y_i,z_i)\,\Delta x \Delta y \Delta z.$$

We evaluate triple integrals by integrating three times from the inner integral to the outer integral as we did with double integrals.

Calculate $\int_0^1 \int_0^z \int_0^y zy^2\,dx\,dy\,dz.$

How to switch the order of the integration? You can draw a 3D picture, or you can switch two integrals at a time. I like the latter technique. As time allows, switch to as many of the six possible combinations of limits of integration.