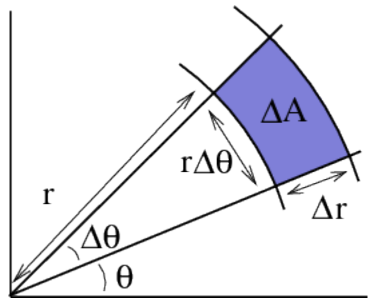
Polar Coordinates and Integration (HW #6)

Most regions of integration are hard to delineate; however, some that are difficult to delineate using rectangular coordinates become easy using the polar coordinate substitution. Substituting into the integrand is easy since $f(x,y) = f(r\cos(\theta), r\sin(\theta))$, but we also must find a substitution for $dx\,dy$.



For very small $\Delta\theta$ and Δr we have

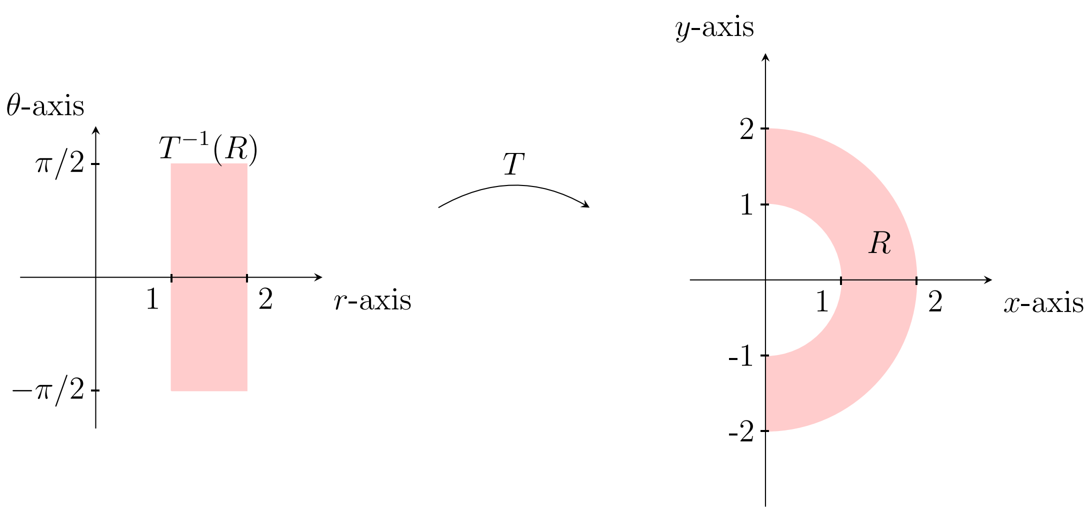
$$\Delta x\,\Delta y = \Delta A \approx r\Delta\theta\,\Delta r.$$

Using Riemann sums and passing to a limit we have

$$\iint\limits_{R} f(x,y)\,dx\,dy = \iint\limits_{T^{-1}(R)} f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta.$$

This identity is written quickly as

$$dx\,dy = r\,dr\,d\theta.$$



For instance, if R is the semi-annular region shown above, then

$$\iint\limits_{R} f(x,y)\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_{1}^{2} f(r\cos(\theta), r\sin(\theta)\,r\,dr\,d\theta.$$

Find $I = \iint\limits_R 1 - x^2 - y^2 \, dA$ if $R : x^2 + y^2 \leq 2x$.

Find $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$.

Find the volume of the part of the solid bounded by $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$ that lies in the first octant.

Applications

The **average value** of $f(x,y)$ over $R = \bar{f} = \dfrac{1}{Area(R)} \iint\limits_R f(x,y)\,dA$ since then $\bar{f}Area(R)$ equals the integral.

If $\delta(x,y)$ is the density of R at each of its points (x,y), then the **mass of** R is $\iint\limits_R \delta\,dA$.

The **weighted average** of $f(x,y)$ over R with density δ is $\dfrac{1}{Mass(R)} \iint\limits_R f(x,y)\delta(x,y)\,dA$.

In second semester calculus you learned that if $f(x,y) = x$, then the **moment about the** y**-axis** is $M_y = \iint\limits_R x\delta(x,y)\,dA$, albeit with a single integral. The formula for M_x is similar: $M_x = \iint\limits_R y\delta(x,y)\,dA$ because we sum up rectangles with very small height and width; in your last calculus class you summed up very thin rectangles with height that did not approach zero and used a much more difficult formula. If we take the weighted average of each moment, we get the **center of mass**

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{Mass(R)}, \frac{M_x}{Mass(R)}\right).$$

The center of mass is the **balancing point** in physics, and the **expected value** of (x,y) in probability where $\dfrac{\delta}{Mass}$ is the **probability density function**.

Find the center of mass of the region R: $1 \le x^2 + y^2 \le 4$ for $x \ge y \ge 0$ if the density is $\delta(x,y) = \dfrac{xy}{(x^2+y^2)^2}$.

Moment of Inertia

mass $= \frac{F}{a}$, the difficulty of imparting a translation to an object. The **moment of inertia** measures the difficulty of imparting a rotation about the:

1) x-axis $= I_x$; 2) y-axis $= I_y$; 3) origin $= I_0$.

For a particle a translation has kinetic energy K.E. $= \frac{mv^2}{2}$. Let angular velocity be $\omega = \frac{d\theta}{dt}$. Since the parameterization of a circle of radius R is $\vec{s}(t) = R\langle\cos(\omega t), \sin(\omega t)\rangle$, rotational velocity is $\|\vec{s}\,'(t)\| = \omega R$. This implies Rotational K.E. is $\frac{mv^2}{2} = \frac{mr^2\omega^2}{2}$. mr^2 takes the place of m in the translation K.E., so we say that mr^2 **is the moment of inertia**. This says it is harder to stop a particle that is more massive and further from the axis of rotation.

Summing up the moment of inertia for each particle gives the moment of inertia for the collection of particles making the region R. $\delta\, dA$ represents the mass for one very small area.

$$I_x = \iint_R y^2 \delta\, dA$$

$$I_y = \iint_R x^2 \delta\, dA$$

$$I_0 = \iint_R (x^2 + y^2)\delta\, dA.$$

Find I_0 for the disk $r \leq a$ if $\delta = 1$.

Now find I_0 for $r = 2a\cos(\theta)$, $\delta = 1$, the same disk as the last example translated right. Decide which disk should have a greater moment of inertia about the origin, and then calculate the moment of inertia for this example to verify your intuition.

As time allows, discuss why the aeorobie (disk with a hole in the middle) of the same weight as a disk without a hole will travel farther if the same force is exerted on either one.

As time allows, discuss probability. Is $f(x, y) = e^{-x^2-y^2}$ a probability density function? Change it so that it is; that is, the area under its graph must be one if integrated over the entire plane.