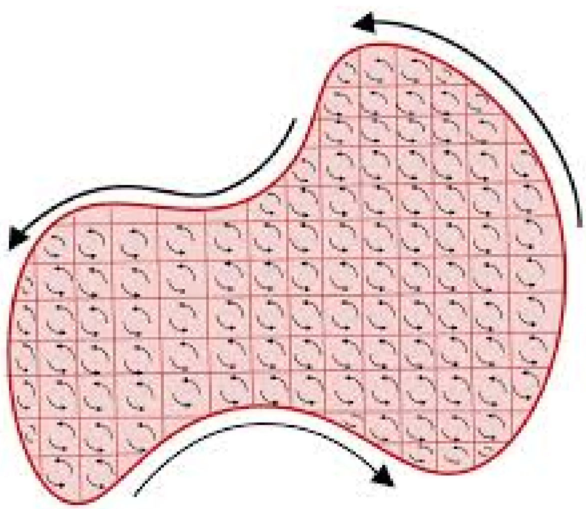
Green's Theorem(HW #6)

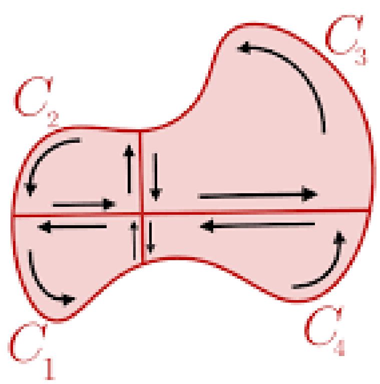
If $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is continuously differentiable, then **Green's Theorem** states the following identity is true. The circle about the line integral tells us the curve is closed and **oriented by the right-hand rule**.

$$\iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \oint_{Bd(R)} \vec{F} \cdot d\vec{s}.$$

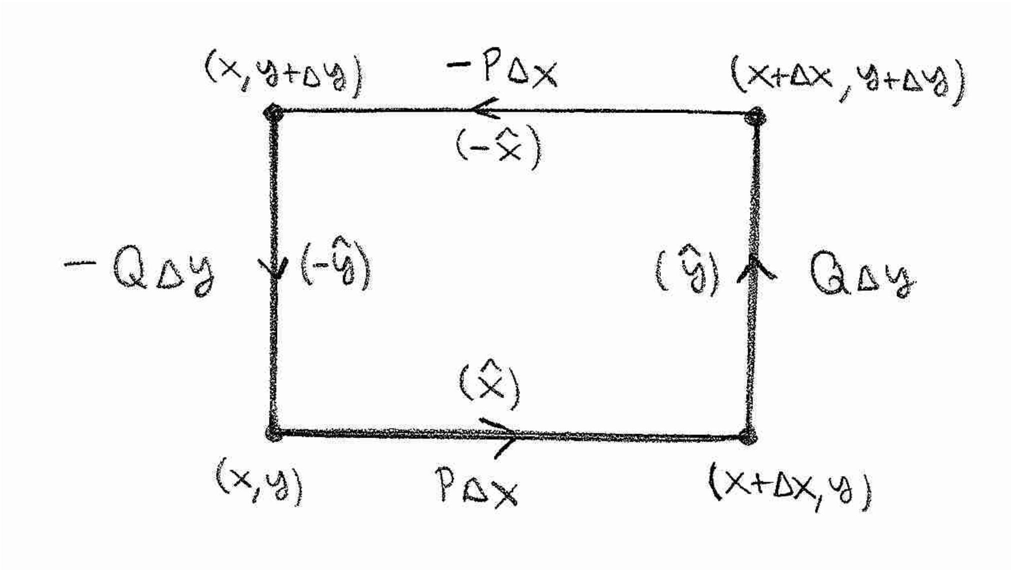
How can we remember this identity? Recall a double integral is a limit of Riemann sums and consider the following:



If we sum line integrals of $\vec{F}$ about each rectangle in the partition oriented by the right-hand rule, then the inner edges cancel and we get $\oint_{Bd(R)} \vec{F} \cdot d\vec{s}$. Notice below how the edges of adjacent regions have opposite orientation.



Since the sum of the line integrals about each rectangle will approach a double integral, I need only convince myself the line integral about one rectangle approaches $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\Delta A$. Consider the picture below. The person who made this picture used $\hat{x} = \hat{\imath}$ and $\hat{y} = \hat{\jmath}$.



The work done by $\vec{F}$ about the curve is approximately

$$[Q(x+\Delta x, y)\Delta y - Q(x,y)\Delta y] - [P(x, y+\Delta y)\Delta x - P(x,y)\Delta x]$$

$$= [Q(x+\Delta x, y) - Q(x,y)]\,\Delta y \cdot \frac{\Delta x \Delta y}{\Delta x \Delta y} - [P(x, y+\Delta y) - P(x,y)]\,\Delta x \cdot \frac{\Delta x \Delta y}{\Delta x \Delta y}$$

Canceling gives:

$$= \left(\frac{Q(x+\Delta x, y) - Q(x,y)}{\Delta x} - \frac{P(x, y+\Delta y) - P(x,y)}{\Delta y}\right)\Delta x \Delta y$$

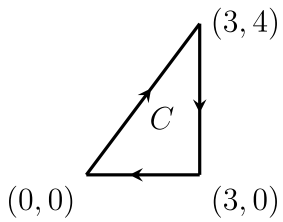
and the sum of these as the norm of the partition approaches 0 is

$$\iint\limits_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

To repeat, Green's Theorem says the sum of the two-dimensional curls within the a region R equals the work around the boundary:

$$\iint\limits_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \oint\limits_{Bd(R)} \vec{F} \cdot d\vec{s}.$$

Use Green's Theorem to find $I = \int\limits_C (2x^2 - 4y)\,dx + \left[5(y+1)^{-3} + 3x\right]\,dy$ if C is given below. Be careful with the orientation!



Use Green's Theorem to find the area inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

As time allows.

Use Green's Theorem to find $I = \oint_C xe^{-2x}\,dx + (3x+y)\,dy$ if $C : 4x^2 + 9y^2 = 36$.

Verify Green's Theorem if $\vec{F} = \langle x+y, x^2 \rangle$ and R is the region $1 \le r \le 2$.

As time allows.

Find the area inside the curve $C : \vec{s}(t) = \langle 1 - \cos(t), \sin(t) + \cos(t) \rangle$.