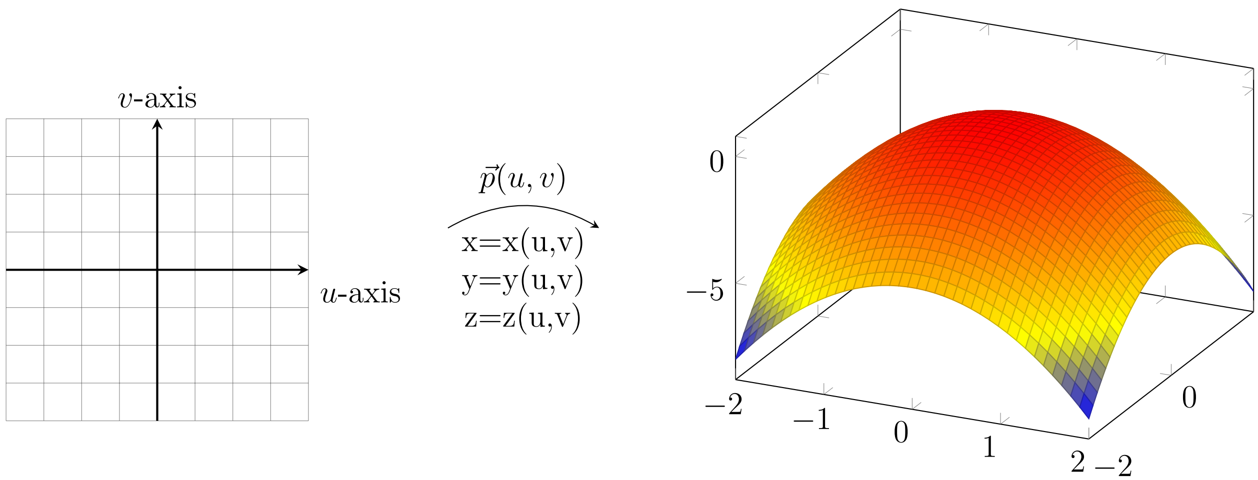
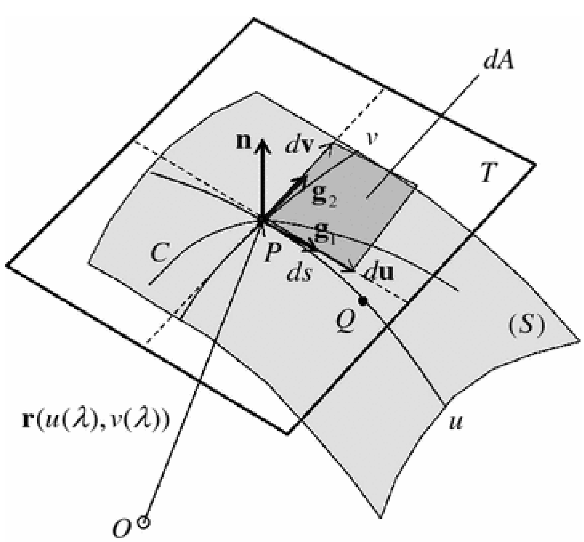
Surface Integrals(HW #7)

Just like line integrals, surface integrals require a parameterization and have two varieties: scalar surface integrals, which calculate mass, and vector surface integrals, which calculate flux. We assume all surfaces are orientable[1].

Parameterize the surface $z = -x^2 - y^2$. Notice the parameterization is a mapping from two-space to three-space.



If we hold one of the parameters constant, then $\vec{p}$ becomes a vector function with derivative. If we scale the two derivative vectors, they determine a very small parallelogram that is a good estimate for the surface near the point of tangency. Here is a picture from the internet that needs a bit of touch-up.



[1]A mobius band is an example of a surface that is not orientable.

If we estimate the area of the parallelogram with the norm of the cross product and then sum the areas from all the parallelograms over a partition, we get an estimate for the area that gets better as the norm of the partition approaches zero. If we dot the cross product with a vector field we get the flux through the parallelogram. If we sum up all such dot products we get the total flux of the vector field through the surface. This motivates our definition of the **surface differential**:

$$d\vec{S} = \pm \vec{p}_u \times \vec{p}_v du\, dv$$

where the sign depends upon the given orientation of the surface. Other notation is $d\vec{S} = d\vec{A} = d\vec{\sigma}$.

Then $\vec{F} \cdot d\vec{S} =$ flux across small parallelogram, so

$$\iint \vec{F} \cdot d\vec{S} = \text{total flux.}$$

For scalar surface intergrals we use

$$dS = \left\| d\vec{S} \right\|$$

so if $\delta(x, y, z)$ represents density, then

$$\iint \delta dS = \text{total mass.}$$

Find the area of the surface parameterized by $p(u, v) = \langle u - v, uv, u + v \rangle$ for $u^2 + v^2 \leq 1$. Notice we don't need an orientation when finding area since area of one side of an orientable surface equals area of the other side.

Find the flux of $\vec{F} = \langle 1, y^2, -1 \rangle$ through the surface $S : \vec{p}(u, v) = \langle u - v, uv, u + v \rangle$, oriented by a positive y-coordinate, for $u^2 + v^2 \leq 1$.

Finding the cross product of a parameterization takes time. There are a few special cases that are used a lot, and so we discuss for these cases how to find the surface differential without much work.

Functions $z = f(x, y)$

We use the parameterization $\vec{p}(u, v) = \langle u, v, f(u, v)\rangle$. Since $x = u$ and $y = v$, we could have just as well written our parameterization as $\vec{p}(x, y) = \langle x, y, f(x, y)\rangle$. Then $\vec{p}_u = \vec{p}_x = \langle 1, 0, f_x(x, y)\rangle$ and $\vec{p}_v = \vec{p}_y = \langle 0, 1, f_y(x, y)\rangle$. The cross product then gives

$$d\vec{S} = \pm\langle -f_x, -f_y, 1\rangle dx\, dy.$$

Verify this.

Find the mass of the surface of $z = x^2 + y^2$ for $x^2 + y^2 \leq 4$ if $\delta(x, y, z) = z - x^2$.

The surface differential is perpendicular to the surface at the point at which it is calculated. That makes finding $d\vec{S}$ easy for coordinate planes. What is $d\vec{S}$ for $z = 3$, $x = 5$, and $y = 0$? If you need more than a picture, notice that all three planes are functions, so you can use the function special-case formula. For instance, $y = y(x, z)$.

Find the flux of $\vec{F}(x, y, z) = \langle x, y, z \rangle$ **out** of the surface of the solid that is bounded by $z = x^2 + y^2$ and $z = 9$. You will need to add the flux from two different surfaces.

Spheres

Using spherical coordinates we parameterize the entire sphere of radius R using

$$\vec{p}(\theta, \phi) = \langle R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi)\rangle.$$

The cross product gets ugly and there is a better way to get the formula: reason geometrically.

First notice the normal to a sphere at a point is the radial vector to that point. Set $\hat{n} = \dfrac{\langle x, y, z\rangle}{R}$ Then

$$d\vec{S} = \pm \hat{n} dS.$$

Secondly, recall the volume differential for spheres is $dV = \rho^2 \sin(\phi) d\rho\, d\phi\, d\theta$. But $\rho = R$ is constant if we are on the sphere so we don't change ρ and get

$$dS = R^2 \sin(\phi) d\phi\, d\theta.$$

Find the flux of $\vec{F}(x, y, z) = \langle y, -x, z\rangle$ out of the piece of the sphere $\rho = 2$ if $0 \leq \phi \leq \dfrac{3\pi}{4}$.

(Notice this surface is not the graph of a function. If $\phi \leq \pi/2$ then we could use the function case with $z = \sqrt{4 - x^2 - y^2}$. Personally I think $d\vec{S} = \pm\hat{n} dS$ is best.)

Cylinders

Using geometry once again, the cylinder $x^2 + y^2 = R^2$, which is *not* a function, has a parameterization $\vec{p}(\theta, z) = \langle R\cos(\theta), R\sin(\theta), z\rangle$ and a surface differential of

$$d\vec{S} = \pm\hat{n}dS$$

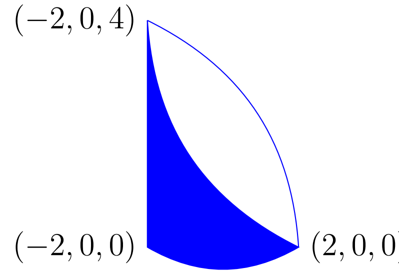
where

$$\hat{n} = \frac{\langle x, y, 0\rangle}{R}$$

and

$$dS = Rd\theta\, dz.$$

Find the mass of the part of the cylinder $x^2 + y^2 = 4$ that lies between $z = 0$ and $x + z = 2$ if the density is $\delta(x, y, z) = z + 4$ grams/meter^2.



Only if time allows.
Find the flux of $\vec{F} = \langle y, x, z \rangle$ out of the tetrahedron with vertices at $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$.