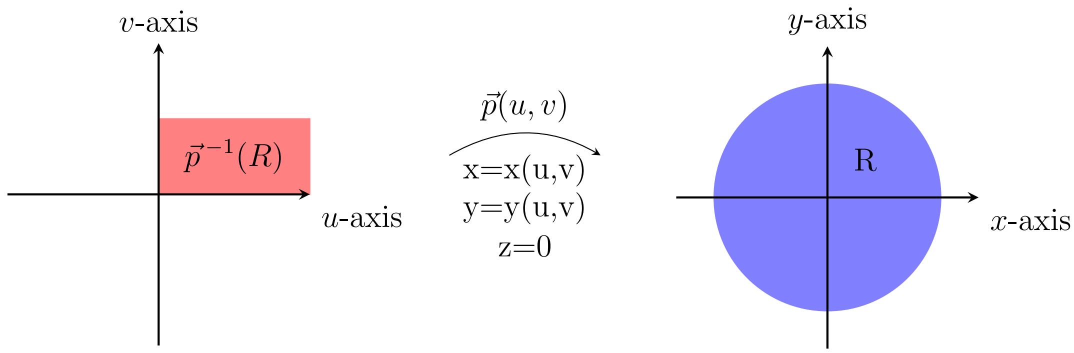
Substitution and the Jacobian(HW #7)

Substitution for double integrals is a scalar surface integral using the parameterization $x = x(u, v)$, $y = y(u, v)$, and $z = 0$.



Then $\vec{p}_u \times \vec{p}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} = \langle 0, 0, x_u y_v - x_v y_u \rangle$ so that

$$dx\,dy = dS = |x_u y_v - x_v y_u|\;du\,dv = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv$$

$\dfrac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u$ is the **Jacobian determinant** of the parameterization and

$$\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_{\vec{p}^{-1}(R)} f(x(u,v), y(u,v))\,\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv.$$

Notice that $\dfrac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u$ equals the 2 x 2 determinant

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \nabla x \\ \nabla y \end{vmatrix}.$$

The **Jacobian matrix** is $\begin{bmatrix} \nabla x \\ \nabla y \end{bmatrix}$, known as the total derivative of $\langle x(u,v), y(u,v)\rangle$.

A familiar example is polar coordinates. We use $x = r\cos(\theta)$ and $y = r\sin(\theta)$ so that

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r.$$

Then $dx\,dy = r\,dr\,d\theta$ so that r is the warping factor that allows us to equate one unit of area in the xy-plane with a unit of area in the $r\theta$-plane.

Find $\iint_R x\,dA$ using the substitution $x = 4u + v$ and $y = u + 4v$ if R is the parallelogram in the first quadrant bounded by $x - 4y = 0$, $x - 4y = -15$, $4x - y = 0$, and $4x - y = 15$.

Hint: $x - 4y = (4u + v) - 4(u + 4v) = -15v$.

Typically you would need to decide what u and v should be for yourself. I would have chosen $u = x - 4y$, $-15 \le u \le 0$ and $v = 4x - y$, $0 \le v \le 15$. Would we get the same result? Notice finding the inverse of R is much easier now, but substituting for x is not: $u - 4v = -15x$ helps.

We also need to be careful calculating the Jacobian. In Linear Algebra we will prove the determinant of an inverse matrix is the reciprocal of the original matrix: $|A|^{-1} = \left|A^{-1}\right|$. Consequently,

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\left(\frac{\partial(x,y)}{\partial(u,v)}\right)}.$$

This is analogous to $\dfrac{dx}{dy} = \dfrac{1}{\left(\frac{dy}{dx}\right)}$ when the derivative is not zero.

Find the mass of the lamina R with density $\delta(x, y) = 1/x^2$ if it is the region in the first quadrant of the xy-plane bounded by $y = x$, $y = 2x$, $y = 1/x$, and $y = 4/x$.

The Jacobian can be generalized to any dimension. Cylindrical coordinates use the transformation $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$. What is the Jacobian matrix? What is $\left|\dfrac{\partial(x, y, z)}{\partial(r, \theta, z)}\right|$?

Spherical coordinates use the transformation $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, and $z = \rho\cos(\phi)$. What is the Jacobian matrix? What is $\left|\dfrac{\partial(x, y, z)}{\partial(r, \theta, z)}\right|$?