The Curl and Stokes' Theorem (HW #8)

If $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$, then we have defined the curl of $\vec{F}$ to be

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle.$$

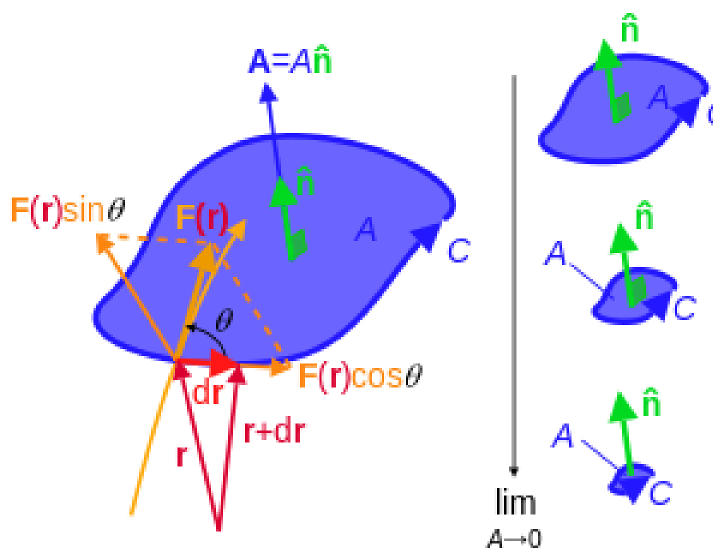
Find the curl of $\vec{F}(x,y,z) = \langle xe^{-y}, xz, ze^{y}\rangle$.

What does this vector mean? There is an equivalent "coordinate free" definition that defines each component of the curl as a **circulation density**.

$\nabla \times \vec{F}(P)$ is the vector defined for the point P such that

$$\nabla \times \vec{F}(P) \cdot \hat{n} = \lim_{\operatorname{diam} C \to 0} \frac{\oint_C \vec{F} \cdot d\vec{s}}{\text{Area in } C}$$

where C is a closed curve in the plane containing P perpendicular to the unit vector $\hat{n}$.



Suppose P is a point in the plane $2x+2y+z=4$ and C is a small closed curve enclosing an area of 0.01 square units on the plane with P inside it. Estimate $\oint_C \vec{F}\cdot d\vec{s}$ if $\nabla\times\vec{F}(P)=\langle 1,2,3\rangle$.

Stokes' Theorem

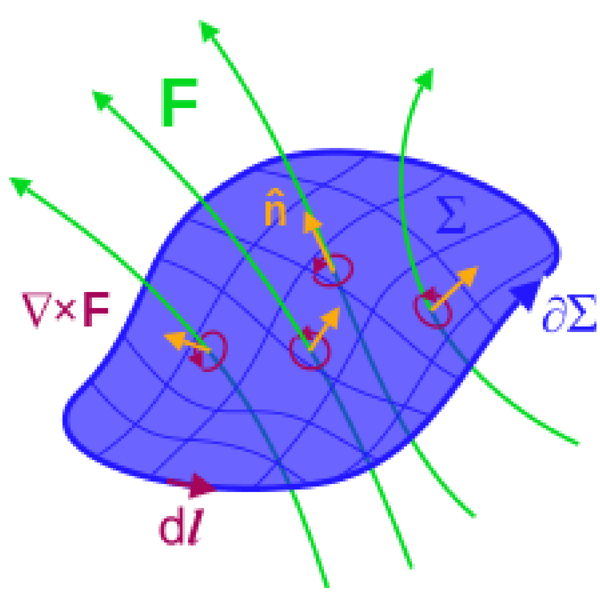
Recall that Green's Theorem is

$$\iint_R Q_x - P_y\, dA = \oint_{Bd(R)} \vec{F}\cdot d\vec{s}.$$

Stokes' Theorem generalizes Green's Theorem from a region in 2-space to a surface in 3-space:

$$\iint_S \nabla\times\vec{F}\cdot d\vec{S} = \oint_{Bd(S)} \vec{F}\cdot d\vec{s}.$$

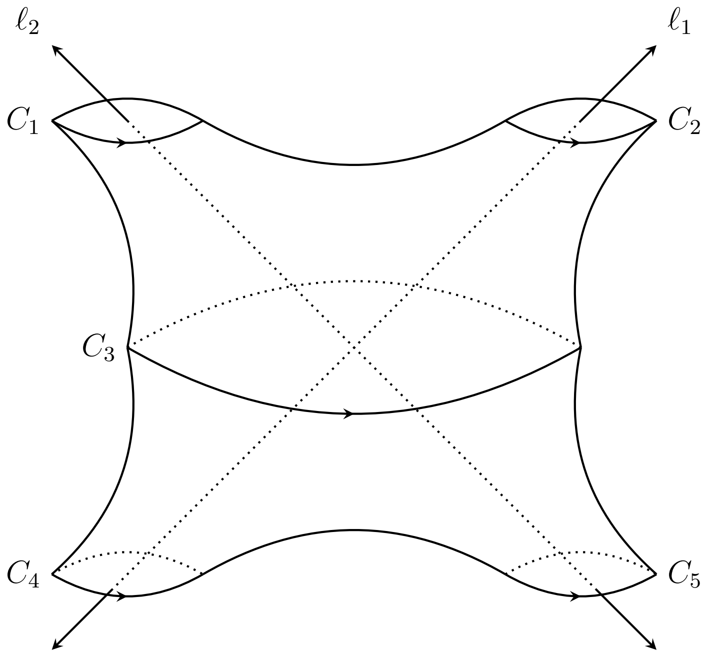
Using the circulation density definition of curl, we can remember the statement of the theorem just as we did Green's theorem: The sum of the curls on the surface equals the line integral about the boundary.



Verify Stokes' Theorem if $\vec{F}(x, y, z) = \langle z, x, yz \rangle$ and $S : x^2 + z^2 \leq 1$ in the plane $y = 2$ oriented in the direction of $\hat{\jmath}$.

Use Stokes' theorem (twice!) to find $I = \iint\limits_S \nabla \times \vec{F} \cdot d\vec{S}$ if $S : z = x^2 + y^2$ for $z \leq 1$ oriented down and if $\vec{F}(x, y, z) = \langle z, x, yz \rangle$. Be careful to find the correct orientation on the boundary.

Double Pants Problem



Given that $\nabla \times \vec{F} = \vec{0}$ everywhere on its domain $\mathbb{R}^3 \setminus (\ell_1 \cup \ell_2)$, $\displaystyle\int_{C_1} \vec{F} \cdot d\vec{s} = 3$, $\displaystyle\int_{C_2} \vec{F} \cdot d\vec{s} = 1$, and $\displaystyle\int_{C_4} \vec{F} \cdot d\vec{s} = 5$, find $\displaystyle\int_{C_3} \vec{F} \cdot d\vec{s}$ and $\displaystyle\int_{C_5} \vec{F} \cdot d\vec{s}$.