The Divergence Theorem (HW #8)

If $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$, then we have defined the divergence of $\vec{F}$ to be

$$\nabla \cdot \vec{F} = P_x + Q_y + R_z.$$

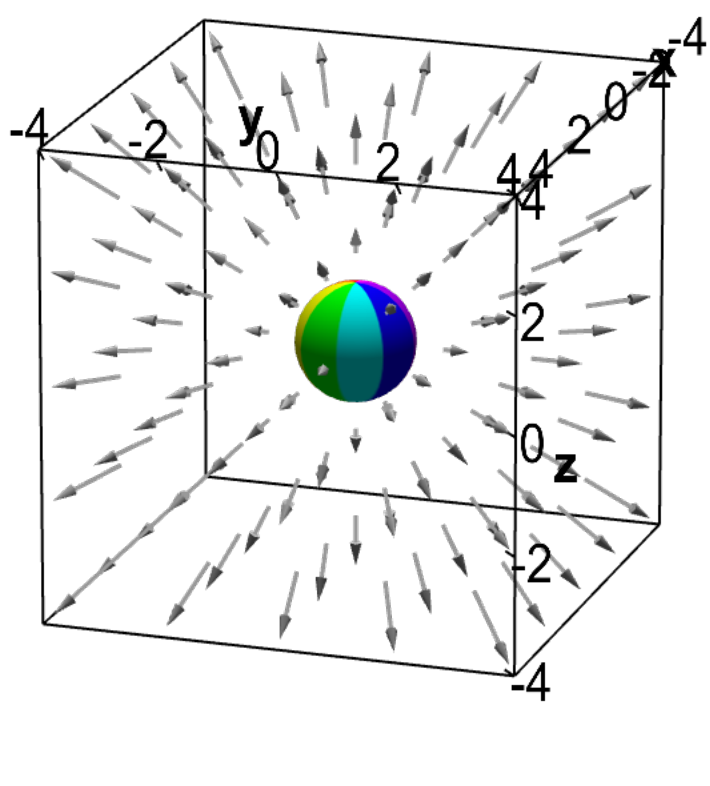
Find the divergence of $\vec{F}(x,y,z) = \langle xe^{-y}, xz, ze^{y}\rangle$.

What does this scalar mean? There is an equivalent "coordinate free" definition that defines divergence as a **flux density**.

$\nabla \cdot \vec{F}(P)$ is the scalar defined for the point P such that

$$\nabla \cdot \vec{F}(P) = \lim_{\operatorname{diam} S \to 0} \frac{\oiint\limits_S \vec{F} \cdot d\vec{S}}{\text{Volume in } S}$$

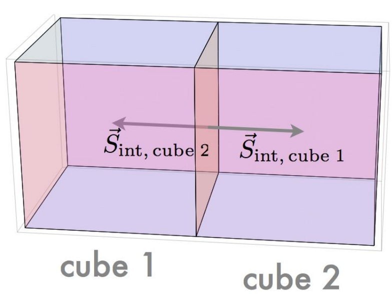
where S is a closed surface oriented outward.

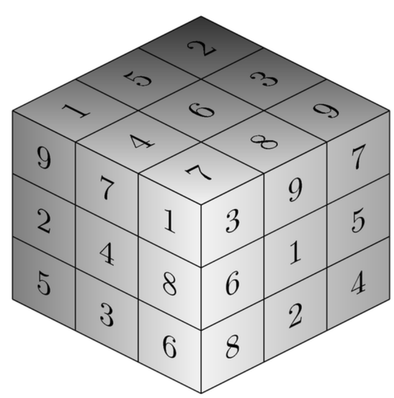


Suppose $\nabla \cdot \vec{F}(P) = 2$ and E is a cube with side length 0.1 μm that contains P. Estimate $\displaystyle\oiint_{Bd(E)} \vec{F} \cdot d\vec{S}$.

Divergence Theorem

The flux out of adjacent cubes cancel on the common edge since the magnitude of the flux is the same, but the orientations are opposite. Hence only the flux through the boundary is left.





Using the flux density definition of curl, we can remember the statement of the theorem just as we did Green's theorem: The sum of the fluxes on a partition of a solid oriented outward equals the surface integral about the (closed) boundary of the solid oriented outward.

The Divergence Theorem

$$\iiint_E \nabla \cdot \vec{F}\, dV = \oiint_{Bd(E)} \vec{F} \cdot d\vec{S}$$

Use the Divergence Theorem to find the flux of $\vec{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle$ out of the boundary of the solid $E : 0 \leq z \leq \sqrt{1 - y^2}$ for $0 \leq x \leq 2$.

Use the Divergence theorem to evaluate $I = \iiint\limits_T 2xyz\,dV$ if T is the solid tetrahedron with intercepts of 1.

Prove as many of the following statements as time allows.

1) $I = \oiint\limits_{Bd(E)} \vec{K} \cdot \hat{n} \, dS = 0$ whenever $\vec{K}$ is a constant vector field.

2) The volume of a solid region E is $\dfrac{1}{3} \oiint\limits_{Bd(E)} \langle x, y, z \rangle \cdot d\vec{S}$.

3) $I = \oiint\limits_{Bd(E)} (\nabla \times \vec{F}) \cdot \hat{n}\, dS = 0.$ Hint: $d\vec{S} = \hat{n} dS.$

4) $I = \oiint\limits_{Bd(E)} \dfrac{\partial f}{\partial \hat{n}} dS = \iiint\limits_{E} \nabla^2 f\, dV.$ Hint: $\dfrac{\partial f}{\partial \hat{n}} = D_{\hat{n}} f.$