Optimization and the Second Partials Test (HW #9)

Recall the Taylor Series about $x = a$.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + E$$

where $|E| << \left|\frac{f''(a)}{2}(x-a)^2\right|$ as $x \to a$.

$f(x)$ has a critical point at $x = a$ if a is in the domain of f and $f'(a) = 0$ or $f'(x)$ does not exist at $x = a$. If $f'(a) = 0$ and f is smooth about an interval of a, how do we know if $f(a)$ is a maximum or a minimum? If x is close to a, then the second term in the Taylor series is 0 shows $f(a)$ is a minimum if $f''(a) > 0$ and a maximum if $f''(a) < 0$. We can't make a conclusion if $f''(a) = 0$. This is the second derivative test. This test can be generalized to multivariable functions.

$P = (a, b)$ is a **critical point** of $f(x, y)$ if P is in the domain of f and f is differentiable at P with $f_x(P) = 0 = f_y(P)$ or f is not differentiable at P.

The second degree two-dimensional Taylor Series about P is

$$f(x,y) = f(P) + f_x(P)(x-a) + f_y(P)(y-b) + \frac{f_{xx}(P)}{2}(x-a)^2 + f_{xy}(P)(x-a)(y-b) + \frac{f_{yy}(P)}{2}(y-b)^2 + E$$

where $|E| << \left|\frac{f_{xx}(P)}{2}(x-a)^2 + f_{xy}(P)(x-a)(y-b) + \frac{f_{yy}(P)}{2}(y-b)^2\right|$.

If f is differentiable in a box about P, and P is a critical point, the linear terms in the Taylor series are zero and $f(P)$ is a maximum if the sum of the quadratic terms are always negative and a minimum if the sum is always positive in a box about P.

Completing squares gives us

$$\Delta = \frac{f_{xx}(P)}{2}(x-a)^2 + f_{xy}(P)(x-a)(y-b) + \frac{f_{yy}(P)}{2}(y-b)^2$$

$$= \frac{1}{2}f_{xx}(P)\left[\left((x-a) + \frac{f_{xy}(P)(y-b)}{f_{xx}(P)}\right)^2 + \left(\frac{f_{xx}(P)f_{yy}(P) - f_{xy}^2(P)}{f_{xx}^2(P)}\right)(y-b)^2\right].$$

$D = f_{xx}(P)f_{yy}(P) - f_{xy}^2(P)$ is called the **discriminant** or **Hessian determinant**. If $D > 0$ then $f(P)$ is a minimum if $f_{xx}(P) > 0$ and a maximum if $f_{xx}(P) < 0$. If $D < 0$ then P is a **saddle point**[1]. If $D = 0$, no conclusion can be made. This is the **second partials test** for $f(x, y)$.

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[1]If $D < 0$ and $f_{xx}(P) > 0$ then $y = b \implies \Delta > 0$ and $y = \dfrac{-(x-a)f_{xx}(P)}{f_{xy}(P)} + b \implies \Delta < 0$.

Similarly if $D < 0$ and $f_{xx}(P) < 0$ then $y = b \implies \Delta < 0$ and $y = \dfrac{-(x-a)f_{xx}(P)}{f_{xy}(P)} + b \implies \Delta > 0$.

Second Partials Test

Suppose P is a critical point for a smooth $f(x,y)$ in a box about P. Let $D = f_{xx}(P)f_{yy}(P) - f_{xy}^2(P)$. Then:

1) $f(P)$ is a maximum if $D > 0$ and $f_{xx}(P) < 0$.

2) $f(P)$ is a minimum if $D > 0$ and $f_{xx}(P) > 0$.

3) P is a saddle point if $D < 0$.

4) No conclusion can be made if $D = 0$.

Notice that no mention is made about whether the critical point is an absolute or local extreme. You will have to inspect the function to determine this.

Find and classify all critical points for $f(x,y) = y + x + \dfrac{1}{xy}$.

Find and classify all critical points for $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$.

If time allows.

Find and classify all critical points for $f(x, y) = \dfrac{x^3}{3} + xy^2 - 8xy + 3$.