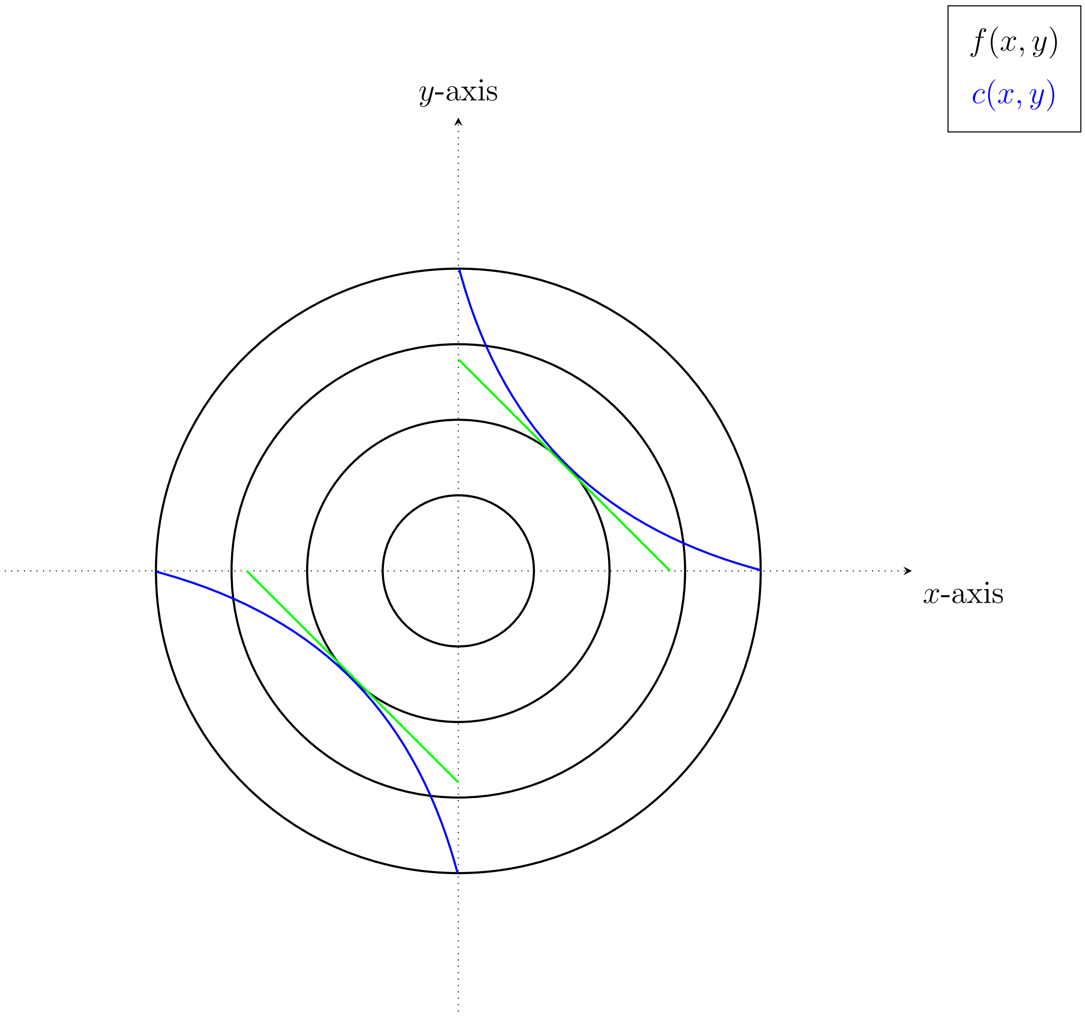
Lagrange Multipliers (HW #9)

Find all points that minimize $f(x, y) = x^2 + y^2$ on the **constraint** $xy = 3$ in the xy-plane. That means we want to find the minimum on only a part of the domain of f. If we draw the contour plot of f and the constraint curve, the solution becomes obvious. Can we generalize a method from this easy problem that can solve more difficult questions?



The level curves show the extreme values of $f(x, y)$ on the constraint are at points for which the two sets of curves share the same tangent line. Consequently, the gradients of $f(x, y)$ and $c(x, y) = xy$ must be parallel at the extreme values. So P is a **constrained critical point** if

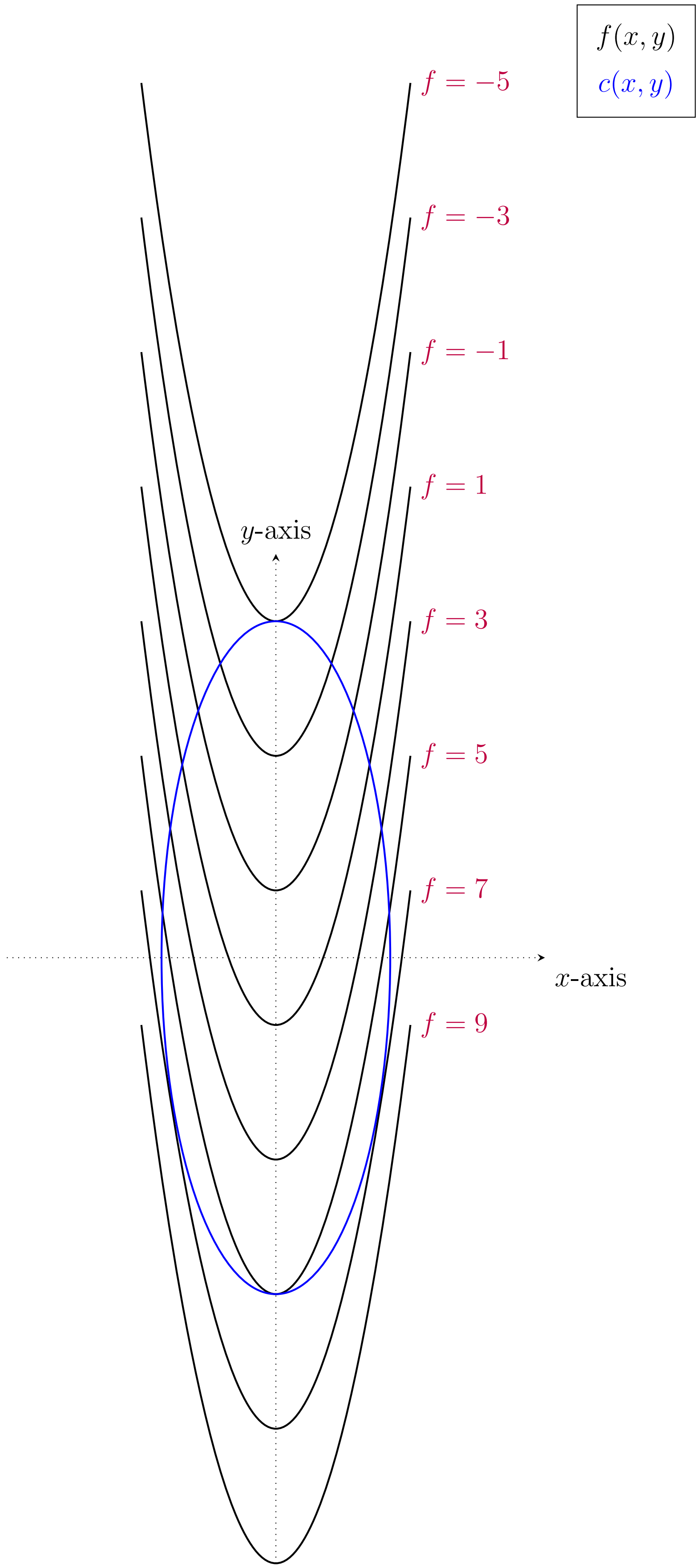
$$\nabla f(P) = \lambda \nabla c(P)$$

for a scalar λ[1].

[1]What if $\lambda = 0$? Then $\nabla f(P) = 0$ and so P is a critical point for the unconstrained domain of f and so also for the constrained domain.

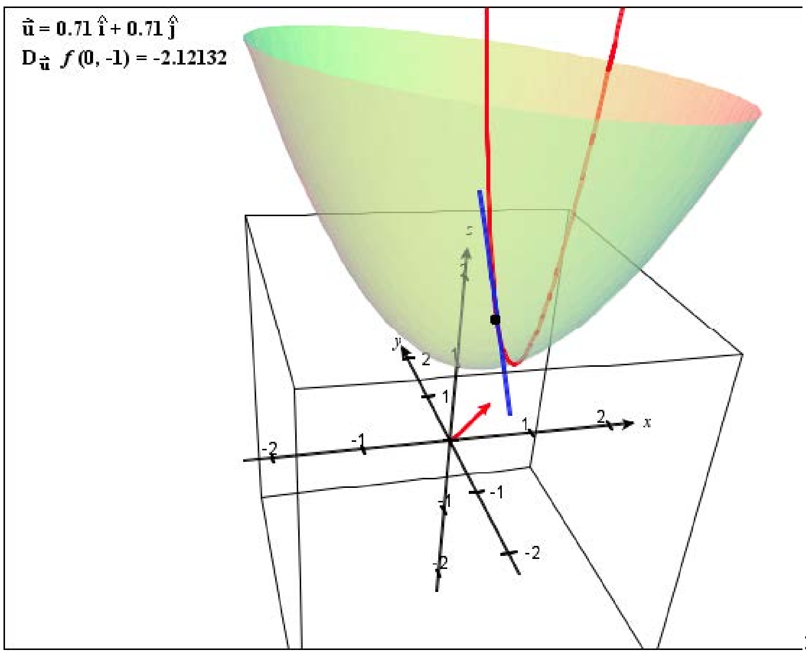
Set up the corresponding system of equations and solve it to get our expected answer.

Find the extrema for $f(x, y) = 2x^2 - y$ if $8x^2 + y^2 = 25$. Use the diagram on the next page to verify your classifications.


$f(x,y)$
$c(x,y)$
$f=-5$
$f=-3$
$f=-1$
$f=1$
$f=3$
$f=5$
$f=7$
$f=9$
y-axis
x-axis

This method generalizes to higher dimensions. The picture below shows the intersection of two constraints, a plane and a paraboloid. The intersection is the red curve; the tangent line to the curve is shown at a point. The gradients of the two constraints determine a plane that includes any vector perpendicular to the tangent line. A point for which the level surface of a function has a gradient perpendicular to the tangent line is a constrained critical point and must then be a combination of the constraint gradients:

$$\nabla f = \lambda_1 \nabla c_1 + \lambda_2 \nabla c_2.$$



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As time allows.

Find and classify all constrained critical points for $f(x, y, z) = \dfrac{x^2 + y^2 + z^2}{2}$ with constraints $z = \dfrac{x^2 + y^2}{2}$ and $z = \dfrac{x^2 - y^2 + 1}{2}$. Next page is left blank for this problem.

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