Quiz 3 25 points Math 200 Name: ______________________

1. Let $f(x, y, z) = (x^2 - y)\cos(\pi z)$. You do not need to repeat work.

(a) (3 points) Find the gradient of $f(x, y, z)$; that is, find ∇f.

$f_x = 2x\cos(\pi z)$

$f_y = -\cos(\pi z)$

$f_z = -\pi(x^2-y)\sin(\pi z)$

$\Rightarrow \boxed{\nabla f = \langle 2x\cos(\pi z), -\cos(\pi z), -\pi(x^2-y)\sin(\pi z)\rangle}$

(b) (3 points) Find and simplify $f_{zz}(x, y, z)$.

$\boxed{f_{zz} = (f_z)_z = -\pi^2(x^2-y)\cos(\pi z)}$

(c) (3 points) Find and simplify $\dfrac{\partial^2 f}{\partial x \partial y}$.

$\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx} = \dfrac{\partial(-\cos(\pi z))}{\partial x} = \boxed{0}$

2. (4 points) Find $\displaystyle\lim_{(x,y)\to(0,2)} \frac{(e^x - 1)(e^y - 1)}{2x}$, or prove that the limit does not exist. Show organized work to defend your answer.

$= \displaystyle\lim_{x\to 0} \frac{e^x-1}{2x} \cdot \lim_{y\to 2}(e^y-1)$

$= \displaystyle\lim_{x\to 0} \frac{e^x}{2} \cdot (e^2-1) = \boxed{\frac{e^2-1}{2}}$

Q3

3. (6 points) Set-up and evaluate an integral that represents the mass of the wire lying on the curve $x = e^y$ from the point $(1, 0)$ to the point $(2, \ln(2))$ if its density is $\delta(x, y) = xe^y$ grams per centimeter.

$\vec{\alpha}(t) = \langle \underbrace{e^t}_{x}, \underbrace{t}_{y} \rangle \Rightarrow \vec{\alpha}'(t) = \langle e^t, 1 \rangle \Rightarrow \|\vec{\alpha}'(t)\| = \sqrt{e^{2t}+1}$

$$\therefore \quad S = \int_0^{\ln(2)} e^t e^t \sqrt{e^{2t}+1}\, dt \quad ; \text{ let } u = e^{2t}+1$$

$$du = 2e^{2t}\,dt \Rightarrow dt = \frac{du}{2(u-1)}$$

$$\Rightarrow S = \int_2^5 \cancel{(u-1)} \sqrt{u} \cdot \frac{du}{2\cancel{(u-1)}}$$

$$= \frac{1}{2} \cdot \frac{2u^{3/2}}{3} \Big|_2^5 = \boxed{\frac{1}{3}(5\sqrt{5} - 2\sqrt{2})} \text{ grams.}$$

4. (6 points) Evaluate $I = \int_C y^2 x\, dx + 2\, dy$ if C is the trace of $\vec{\alpha}(t) = \langle e^t, e^{-t} \rangle$ for $0 \le t \le 1$.

$x \quad y$

$\vec{\alpha}'(t) = \langle e^t, -e^{-t} \rangle$

(note: $y^2 x\, dx + 2\, dy = \langle y^2 x, 2 \rangle \cdot \langle dx, dy \rangle$.)

$$\Rightarrow I = \int_0^1 \langle e^{-2t} e^t, 2 \rangle \cdot \langle e^t, -e^{-t} \rangle\, dt$$

$$= \int_0^1 1 - 2e^{-t}\, dt$$

$$= t + 2e^{-t} \Big|_0^1$$

$$= \left(1 + \frac{2}{e}\right) - 2 = \boxed{\frac{2}{e} - 1}$$