Quiz 5 25 points Math 200 Name: ______________________

1. (7 points) Find the constants a and b that make $\vec{F}$ a gradient field and then use the FTCLI to find $\int_C \vec{F} \cdot d\vec{s}$ if $\vec{F}(x,y) = \langle \underbrace{y + \cos(\pi y)}_{\Phi_x}, \underbrace{ax + bx\sin(\pi y)}_{\Phi_y} \rangle$ and $C : \vec{\alpha}(t) = \langle 2\cos(t), 3\sin(t) \rangle$ for $0 \le t \le \pi$.

$\Phi_{xy} = 1 - \pi \sin(\pi y)$

$\Phi_{yx} = a + b\sin(\pi y)$. $\Phi_{xy} = \Phi_{yx} \Rightarrow \boxed{a = 1,\ b = -\pi.}$

$$\Phi = \int \Phi_x \, dx = xy + x\cos(\pi y) + f(y)$$

Then $x - \pi x \sin(\pi y) = \Phi_y = \dfrac{\partial(xy + x\cos(\pi y) + f(y))}{\partial y} = x - \pi x \sin(\pi y) + f'(y)$

$\Rightarrow f'(y) = 0 \Rightarrow f(y) = \text{constant} = C.$ Let $C = 0$.

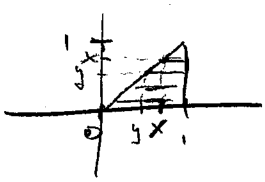
Then $\boxed{\Phi(x,y) = xy + x\cos(\pi y),}$ and

$$\int_C \vec{F} \cdot d\vec{s} \underset{\text{FTCLI}}{=} \Phi(\alpha(\pi)) - \Phi(\alpha(0)) = \Phi(-2,0) - \Phi(2,0)$$

$$= -2 - 2 = \boxed{-4}$$

2. (6 points) Evaluate $I = \displaystyle\int_0^1 \int_y^1 \cos\left(\frac{\pi x^2}{2}\right) dx\, dy$.

Cannot integrate using the given order, so switch the order.



$$\Rightarrow I = \int_0^1 \int_0^x \cos\left(\frac{\pi x^2}{2}\right) dy\, dx$$

$$= \int_0^1 y\cos\left(\frac{\pi x^2}{2}\right)\Big|_0^x dx$$

$$= \int_0^1 x\cos\left(\frac{\pi x^2}{2}\right) dx$$

$$= \frac{1}{\pi}\sin\left(\frac{\pi x^2}{2}\right)\Big|_0^1 = \boxed{\frac{1}{\pi}}$$

Q5

3. (6 points) Calculate $I = \int_0^1 \int_{\cancel{1-x}\ -x}^{\cancel{1+x}\ x} \int_0^{xy} 4z\,dz\,dy\,dx.$

$$I = \int_0^1 \int_{1-x}^{1+x} 2z^2 \Big|_0^{xy} dy\,dx$$

$$= \int_0^1 \int_{-x}^{x} 2x^2y^2\,dy\,dx$$

$$= \int_0^1 2x^2 \frac{y^3}{3} \Big|_{-x}^{x} dx$$

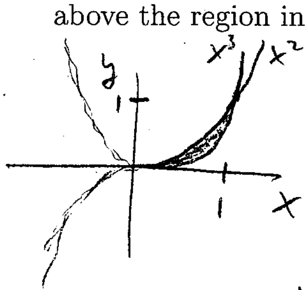
$$= \frac{2}{3} \int_0^1 x^2 \left[x^3 - (-x)^3 \right] dx$$

$$= \frac{4}{3} \int_0^1 x^5\,dx$$

$$= \frac{4}{3} \cdot \frac{x^6}{6} \Big|_0^1$$

$$= \boxed{\frac{2}{9}}$$

4. (6 points) Use a double integral to find the volume of the solid below the graph of $z = 2y + x^2$ and above the region in the xy-plane bounded by $y = x^2$ and $y = x^3$. ~~in the first quadrant.~~



(notice $z \geq 0$ on this region, so net volume = volume in this case.)

$$\text{Vol} = \int_0^1 \int_{x^3}^{x^2} 2y + x^2\,dy\,dx$$

$$= \int_0^1 y^2 + x^2 y \Big|_{x^3}^{x^2} dx$$

$$= \int_0^1 (x^4 + x^4) - (x^6 + x^5)\,dx$$

$$= \frac{2x^5}{5} - \frac{x^7}{7} - \frac{x^6}{6} \Big|_0^1 = \frac{2}{5} - \frac{1}{7} - \frac{1}{6} = \frac{84 - 30 - 35}{210} = \boxed{\frac{19}{210}}$$