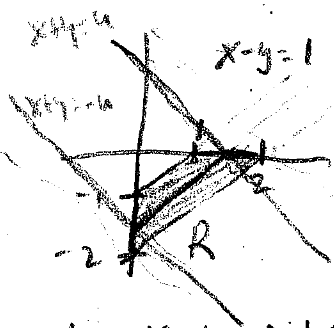
Quiz 7 25 points Math 200 Name: ____________________

1. (7 points) R is the trapezoidal region with vertices at $(1,0)$, $(2,0)$, $(0,-2)$, and $(0,-1)$. Change variables to find $I = \iint_R e^{\frac{x+y}{x-y}}\,dx\,dy$.



$u = x - y \Rightarrow 1 \le u \le 2$

$w = x + y \Rightarrow -u \le w \le u$

$\dfrac{\partial(u,w)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$

$\Rightarrow \left| \dfrac{\partial(x,y)}{\partial(u,w)} \right| = \dfrac{1}{2}.$

$$\therefore I = \int_1^2 \int_{-u}^{u} e^{\frac{w}{u}} \cdot \frac{1}{2}\,dw\,du$$

$$= \frac{1}{2}\int_1^2 u\, e^{\frac{w}{u}} \Big|_{-u}^{u}\, du$$

$$= \frac{1}{2}\int_1^2 u\,[e^1 - e^{-1}]\,du$$

$$= \frac{e - e^{-1}}{4}\left(u^2 \Big|_1^2 \right)$$

$$= \boxed{\frac{3}{4}(e - e^{-1})},$$

2. (6 points) Find the mass of the piece of the cylinder $x^2 + y^2 = \cancel{9}\ 4$ that lies above the plane $z = 0$ and below the plane $z = \cancel{4} + x + \cancel{y}$ if the density is $\delta(x,y,z) = z(x^2 + y^2)$ grams per square centimeter.

$$\text{Mass} = \iint_S \delta\, dS \; ; \quad S \text{ is a piece of a cylinder}$$

$$\Rightarrow dS = 2\,d\theta\,dz \; ; \; \delta = z \cdot 4.$$

$$\therefore \text{Mass} = \int_0^{2\pi}\int_0^{2+2\cos\theta} 4z \cdot 2\,dz\,d\theta$$

$$= \int_0^{2\pi} 8 \cdot \frac{z^2}{2}\Big|_0^{2+2\cos\theta}\, d\theta$$

$$= 4\int_0^{2\pi} 4 + 8\cancel{\cos\theta} + 4\cos^2\theta\, d\theta \quad \overset{0;\,1T}{}$$

$$= 16\int_0^{2\pi} 1 + \frac{1 + \cancel{\cos(2\theta)}}{2}\, d\theta = \boxed{48\pi} \quad \overset{0;\,2T}{}$$

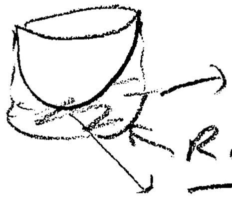
Q7

3. (6 points) Find $I = \iint_S z\, dS$ if S is the part of the paraboloid $z = x^2 + y^2$ that lies underneath the plane $z = 1$.

S is part of the graph for $z = x^2+y^2 \Rightarrow d\vec{S} = \pm\langle -2x, -2y, 1\rangle dx dy$

$\Rightarrow dS = \sqrt{4x^2+4y^2+1}\, dx dy.$

$\left.\begin{array}{l} z = x^2+y^2 \\ 0 \le z \le 1 \end{array}\right\} \Rightarrow$



$R: x^2+y^2 \le 1$

$\therefore\ I = \iint_R (x^2+y^2)\sqrt{4(x^2+y^2)+1}\, dA$

$I = \int_0^{2\pi}\int_0^1 r^2\sqrt{4r^2+1} \cdot r\, dr d\theta$

$\boxed{\begin{array}{l} u = 4r^2+1 \\ du = 8r\,dr \end{array}} \Rightarrow I = 2\pi \cdot \int_1^5 \frac{u-1}{4}\sqrt{u} \cdot \frac{1}{8}\, dr$

$\Rightarrow I = \frac{2\pi}{32}\int_1^5 u^{\frac{3}{2}} - u^{\frac{1}{2}} du$

$= \frac{\pi}{16}\left(\frac{2u^{5/2}}{5} - \frac{2}{3}u^{\frac{3}{2}}\right|_1^5$

$\Rightarrow I = \frac{\pi}{8}\left[\left(5\sqrt{5} - \frac{5\sqrt{5}}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)\right]$

$= \boxed{\frac{\pi}{8}\left[\frac{10\sqrt{5}}{3} + \frac{2}{15}\right]}$

4. (6 points) Find the flux of $\vec{F}(x, y, z) = \langle -z, 3y, x\rangle$ through the sphere $x^2 + y^2 + z^2 = 4$ if it is oriented outward.

S a sphere $\Rightarrow d\vec{S} = \frac{\langle x, y, z\rangle}{2} \cdot 4\sin^2\phi\, d\phi\, d\theta$, oriented out.

$\text{Flux} = \int_0^{2\pi}\int_0^{\pi} \langle -z, 3y, x\rangle \cdot \frac{\langle x, y, z\rangle}{2} \cdot 4\sin\phi\, d\phi\, d\theta$

$= \int_0^{2\pi}\int_0^{\pi} 6(2\sin\phi\sin\theta)^2 \cdot \sin\phi\; d\phi\; d\theta$

$= 24 \cdot \int_0^{2\pi}\sin^2\theta\, d\theta \cdot \int_0^{\pi}\sin\phi(1-\cos^2\phi)\, d\phi$

$= 24 \cdot \int_0^{2\pi}\frac{1 - \cancel{\cos(2\theta)}^{0;2\pi}}{2}\, d\theta \cdot \int_0^{\pi}\sin\phi - \sin\phi\cos^2\phi\, d\phi$

$= 24 \cdot \pi \cdot \left(-\cos\phi\right|_0^{\pi} + \left(\frac{\cos^3\phi}{3}\right|_0^{\pi}$

$= 24 \cdot \pi \cdot \left(2 - \frac{2}{3}\right) = 24\pi \cdot \frac{4}{3} = \boxed{32\pi}$