Test 1 100 points Math 200 Name: ______________________

1. ($\cancel{8}$ 6 points) Find the equation of the plane in standard form that contains the point $P = (1, -1, 2)$, and is parallel to the vectors $\langle 4, 1, -1\rangle$ and $\langle 1, 0, 2\rangle$.

$$\begin{array}{r} \langle 4,1,-1\rangle \\ \times\ \langle 1,0,2\rangle \\ \hline \langle 2,-9,-1\rangle \end{array} \Rightarrow 2x-9y-z = \langle 2,-9,-1\rangle\cdot\langle 1,-1,2\rangle$$

$$\Rightarrow 2x-9y-z = 2+9-2$$

$$\Rightarrow \boxed{2x-9y-z=9}$$

2. Let $f(x, y) = \ln(xy) + e^{xy^2} + 6$. You may use work from one part in another.

(a) ($\cancel{4}$ 6 points) Find the gradient of $f(x, y)$ at $(1, 1)$; that is, find $\nabla f(1, 1)$.

$$\nabla f(1,1) = \left\langle \frac{1}{x}+y^2e^{xy^2},\ \frac{1}{y}+2xy\,e^{xy^2}\right\rangle\Big|_{(1,1)}$$

$$= \boxed{\langle 1+e,\ 1+2e\rangle}$$

(b) (6 points) Find the second partial derivatives of $f(x, y)$.

$$\boxed{f_{xx} = -\frac{1}{x^2}+y^4e^{xy^2}}$$

$$\boxed{f_{yx} = f_{xy} = (2y+2xy^3)e^{xy^2} \text{ or } 2y\,e^{xy^2}(1+xy^2)}$$

$$\boxed{f_{yy} = -\frac{1}{y^2}+e^{xy^2}(2x+4x^2y^2) \text{ or } -\frac{1}{y^2}+2xe^{xy^2}(1+2xy^2)}$$

T1,2

3. Let $\vec{u} = \langle 1, 2, -1\rangle$, $\cancel{\vec{v}}\ \vec{p} = \langle 1, -2, \cancel{3}\ 4\rangle$, and $\vec{w} = 2\hat{\imath} - \hat{\jmath} + 3\hat{k}$. You may use work from one part in other parts.

(a) (4 points) Find the area of the triangle determined by $\vec{u}$ and $\cancel{\vec{v}}\ \vec{p}$

$$\text{Area} = \frac{\|\vec{u} \times \vec{p}\|}{2} = \frac{\sqrt{36+25+16}}{2} = \boxed{\frac{\sqrt{77}}{2}}$$

$$\begin{array}{rl} & \vec{u} = \langle 1, 2, -1\rangle \\ \times & \vec{p} = \langle 1, -2, 4\rangle \\ \hline & \langle 6, -5, -4\rangle \end{array}$$

(b) (4 points) Find the flux of $\vec{w}$ through the parallelogram spanned by $\vec{u}$ and $\cancel{\vec{v}}\ \vec{p}$ and oriented from $\vec{u}$ to $\cancel{\vec{v}}\ \vec{p}$

$$\vec{w} \cdot (\vec{u} \times \vec{p}) = \langle 2, -1, 3\rangle \cdot \langle 6, -5, -4\rangle$$

$$= 12 + 5 - 12$$

$$= 5$$

(c) (4 points) Find $\cancel{\vec{v}}\ \vec{p}_{\|\vec{w}}$.

$$\vec{p}_{\|\vec{w}} = \frac{\vec{p} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} = \frac{\langle 1, -2, 4\rangle \cdot \langle 2, -1, 3\rangle}{4+1+9} \vec{w}$$

$$= \frac{2+2+12}{14} \vec{w} = \boxed{\frac{8}{7} \vec{w} \text{ or } \frac{8}{7}\langle 2, -1, 3\rangle}$$

(d) (4 points) Find $\cancel{\vec{v}}\ \vec{p}_{\perp\vec{w}}$.

$$\vec{p}_{\perp\vec{w}} = \vec{p} - \vec{p}_{\|w} = \langle 1, -2, 4\rangle - \frac{8}{7}\langle 2, -1, 3\rangle$$

$$= \boxed{\frac{1}{7}\langle -9, -6, 4\rangle}$$

T1, 3

4. Let $\vec{\alpha}(t) = \langle \sin(2t), \cos(2t), t \rangle$ for all parts of this problem. You do not need to repeat work.

(a) (8 points) Find the arc length of the trace of $\vec{\alpha}(t)$ from $t = 0$ to $t = \pi$.

$\vec{\alpha}'(t) = \langle 2\cos(2t), -2\sin(2t), 1 \rangle$

$\Rightarrow \|\vec{\alpha}'(t)\| = \sqrt{4\cos^2(2t) + 4\sin^2(2t) + 1} = \sqrt{5}$, so

$ds = \sqrt{5}\,dt$.

Then $s = \int_C ds = \int_0^{\pi} \sqrt{5}\,dt = \boxed{\sqrt{5}\pi}$

(b) (6 points) Find the equation of the osculating plane for the trace of $\vec{\alpha}(t)$ at $t = \pi$. Write the answer in $ax + by + cz = d$ form.

$\vec{v} = \vec{\alpha}'(\pi) = \langle 2, 0, 1 \rangle$; $\quad \vec{\alpha}''(t) = \langle -4\sin(2t), -4\cos(2t), 0 \rangle$

$\times\ \vec{a} = \vec{\alpha}''(\pi) = \langle 0, -4, 0 \rangle \qquad \vec{\alpha}(\pi) = \langle 0, 1, \pi \rangle$

$\langle 4, 0, -8 \rangle$; use $\vec{n} = \langle 1, 0, -2 \rangle$.

Then the osculating plane is

$\langle 1, 0, -2 \rangle \cdot \langle x, y, z \rangle = \langle 1, 0, -2 \rangle \cdot \langle 0, 1, \pi \rangle$

$\Rightarrow \boxed{x - 2z = -2\pi}$

(c) (4 points) Find the curvature for the trace of $\vec{\alpha}(t)$ at $t = \pi$. Simplify your final answer.

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{\|\langle 4, 0, -8 \rangle\|}{(\sqrt{4+1})^3} = \frac{4\sqrt{1+4}}{5\sqrt{5}} = \boxed{\frac{4}{5}}$$

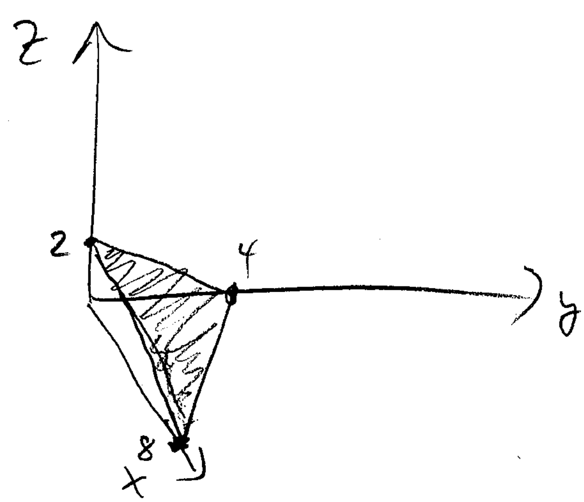
T1,4

5. (6 points) Write a position function with input t seconds for a particle that moves once counterclockwise with respect to $\hat{\jmath}$ around a circle every 2 seconds if the circle is in the plane $y = 4$ with center at $(2, 4, 1)$ and radius 3.

$$\vec{\alpha}(t) = \underbrace{\langle 3\sin(\pi t), 0, 3\cos(\pi t)\rangle}_{\text{(circle)}} + \underbrace{\langle 2, 4, 1\rangle}_{\text{(translation)}}$$

$$\Rightarrow \boxed{\vec{\alpha}(t) = \langle 2 + 3\sin(\pi t), 4, 1 + 3\cos(\pi t)\rangle}$$

6. (4 points) Sketch the part of the plane $x + 2y + 4z = 8$ in the first octant (x, y, and z are all non-negative) with labeled positively oriented xyz-axes.



7. (6 points) Find $\vec{f}\,'(1)$ if $\vec{f}(t) = \vec{p}(t) \times \langle 1, t, t^2\rangle$, $\vec{p}(1) = \hat{\imath} - 2\hat{\jmath} + 2\hat{k}$, and $\left.\frac{d\vec{p}}{dt}\right|_{t=1} = \langle 0, 1, -1\rangle$. Show work to defend your answer.

$$\vec{f}\,'(1) = \vec{p}\,'(1) \times \langle 1, 1, 1\rangle + \vec{p}(1) \times \langle 0, 1, 2t\rangle\Big|_{t=1}$$

$$= \underline{\langle 0, 1, -1\rangle \times \langle 1, 1, 1\rangle} + \underline{\langle 1, -2, 2\rangle \times \langle 0, 1, 2\rangle}$$

$$= \langle 2, -1, -1\rangle + \langle -6, -2, 1\rangle = \boxed{\langle -4, -3, 0\rangle}$$

$\pi, 5$

from (0,0) to (1,1) that lies on $y=x^3$

8. (10 points) Find the mass of the wire that lies on the trace of the curve $\vec{p}(t) = \cancel{\langle t, t^3 \rangle}$ ~~for~~ $\cancel{0 \le t \le 2}$ if the density of the wire is $\delta(x, y) = \sqrt{1 + 9xy}$ grams/cm.

$$\text{Mass} = \int_C \delta \, ds. \qquad \vec{p}(t) = \langle t, t^3 \rangle, \quad 0 \le t \le 1$$

$$\vec{p}\,'(t) = \langle 1, 3t^2 \rangle \Rightarrow \|\vec{p}\,'(t)\| = \sqrt{1+9t^4}.$$

$$\text{Then Mass} = \int_0^1 \sqrt{1+9t\,t^3}\,\sqrt{1+9t^4}\,dt$$

$$= \int_0^1 1+9t^4\,dt$$

$$= t + \frac{9}{5}t^5 \Big|_0^1 = 1 + \frac{9}{5} = \boxed{\frac{14}{5}} \text{ grams}$$

9. Find the limit if possible, or prove that the limit does not exist. Show organized work to defend your answer.

(a) (4 points) $\displaystyle\lim_{(x,y)\to(1,-1)} \cancel{\ln}\left(\frac{1+y}{x^2+xy}\right)$

$$\lim_{\substack{y \to -1^+ \\ x=1}} \frac{1+y}{1+y} = \lim_{y\to -1^+} 1 = \underline{1} \; ; \; \lim_{\substack{x\to 1^+ \\ y=-1}} \frac{0}{1} = \underline{0}$$

$1 \ne 0 \Rightarrow$ the limit $\boxed{\text{DNE}}$

(b) (4 points) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{\sin(\arctan(y))}{x^2+y^2}$

$$= \frac{\sin(\pi/4)}{1} = \boxed{\frac{\sqrt{2}}{2}}$$

T1,6

10. (10 points) Find the work done by the vector field $\vec{F}(x,y,z) = -y\hat{\imath} + z\hat{\jmath} + x\hat{k}$ on a particle that moves on the trace of $\vec{\alpha}(t) = \langle 2+t^{-1}, t^3, t^2 \rangle$ for $0 < t \leq 1$.

$\alpha'(t) = \langle -t^{-2}, 3t^2, 2t \rangle$

$$W = \int_0^1 \langle -t^3, t^2, 2+t^{-1} \rangle \cdot \langle -t^{-2}, 3t^2, 2t \rangle \, dt$$

$$= \int_0^1 t + 3t^4 + 4t + 2 \, dt$$

$$= \frac{3t^5}{5} + \frac{5t^2}{2} + 2t \Big|_0^1$$

$$= \frac{3}{5} + \frac{5}{2} + 2 = \frac{6+25+20}{10} = \boxed{\frac{51}{10}}$$

11. (5 points) Sketch $y = z^2 + x^2$ with labeled positively oriented xyz-axes.