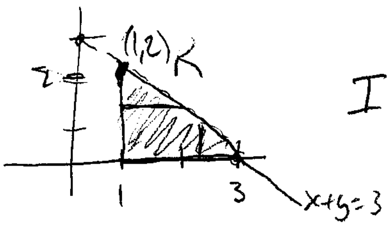
Test 2 100 points Math 200 Name: ______________________

1. (6 points) Sketch the region of integration for $I = \int_1^3 \int_0^{3-x} f(x,y)\,dy\,dx$ and then rewrite I with the order of integration switched.



$$I = \int_0^2 \int_1^{3-y} f(x,y)\; dx\, dy$$

2. (7 points) Find the tangent plane in standard form for the graph of $xz + 2x^2y + y^2z^3 = 11$ at the point $Q = (2,1,1)$.

Let $f(x,y,z) = xz + 2x^2y + y^2z^3$

Then $\nabla f(Q) = \langle z + 4xy,\ 2x^2 + 2yz^3,\ x + 3y^2z^2 \rangle \big|_Q = \langle 9, 10, 5 \rangle$

is $\perp$ to the plane.

$\therefore$ the equation is $9x + 10y + 5z = 18 + 10 + 5$

$\Rightarrow \boxed{9x + 10y + 5z = 33}$

3. (5 points) Approximate $f(2.3, 4.9)$ using a differential if $f(2,5) = 1$ and $\nabla f(2,5) = \langle 3, 8 \rangle$.

$f(2.3, 4.9) \approx f(2,5) + df$

$= 1 + 3(.3) + 8(-.1)$

$= 1 + .9 - .8$

$= \boxed{1.1}$

T2, 2

Use the FTCLI to evaluate the integrals on this page whenever possible. If not possible, show work indicating this, and then evaluate the integral using another ~~way~~ method ?

4. (9 points) $I = \int_C [y + \sin(4y)]\, dx + [x + 4x\cos(4y)]\, dy$ if C is parameterized by $\vec{\alpha}(t) = \langle \sin(t), t\rangle$ for $\pi/2 \le t \le \pi$.

I $\frac{\partial(x+4x\cos(4y))}{\partial x} = 1+4\cos(4y) = \frac{\partial(y+\sin(4y))}{\partial y} \Rightarrow \vec{F} = \nabla\phi.$

II $\phi = \int y+\sin(4y)\, \partial x = x\,(y+\sin(4y)) + f(y)$

$\Rightarrow x\,(1+4\cos(4y)) = \phi_y = x(1+4\cos(4y)) + f'(y) \Rightarrow f(y) = \text{constant}.$

Use $\underline{\phi = x\,(y+\sin(4y))}$. Then

III $I = \phi(\alpha(\pi)) - \phi(\alpha(\tfrac{\pi}{2}))$

$= \phi(0,\pi) - \phi(1,\tfrac{\pi}{2})$

$= 0 - (\tfrac{\pi}{2}+0) = \boxed{-\frac{\pi}{2}}$

5. (9 points) $I = \int_C [y + \sin(4y)]\, dx + [x + x\cos(4y)]\, dy$ if C is parameterized by $\vec{\alpha}(t) = \langle t^3, t\rangle$ for $-\pi/2 \le t \le \pi/2$.

x y

I $\frac{\partial(x+x\cos(4y))}{\partial x} = 1+\cos(4y) \ne 1+4\cos(4y) = \frac{\partial(y+\sin(4y))}{\partial y} \Rightarrow \vec{F} \ne \nabla\phi.$

II $x=t^3,\ y=t$; $x'=3t^2,\ y'=1$ $\Rightarrow I = \int_{-\pi/2}^{\pi/2} (t+\sin(4t))\cdot 3t^2 + [t^3+t^3\cos(4t)]\cdot 1\, dt$

$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \cancel{3t^3}^{\to 0} + \cancel{3t^2\sin(4t)}^{\to 0} + \cancel{t^3}^{\to 0} + \cancel{t^3\cos(4t)}^{\to 0}\, dt$

ODD

$= \boxed{0}.$

T2, 3

6. (9 points) Evaluate $I = \int_0^2 \int_0^y \int_x^1 6xy\cancel{z}\, dz\, dx\, dy$.

$$I = \int_0^2 \int_0^y 6xyz \Big|_x^1 dx\, dy$$

$$= \int_0^2 \int_0^y 6xy - 6x^2y \, dx\, dy$$

$$= \int_0^2 y\left(3x^2 - 2x^3\right)\Big|_0^y dy$$

$$= \int_0^2 3y^3 - 2y^4 \, dy$$

$$= \frac{3y^4}{4} - \frac{2y^5}{5}\Big|_0^2$$

$$= 12 - \frac{64}{5}$$

$$= \boxed{-\frac{4}{5}}$$

7. (5 points) What is the tangent plane to the surface $z = f(x, y)$ at the point $P = (4, -1, 2)$ if $f(4, -1) = 2$ and $\nabla f(4, -1) = \langle 1, -3 \rangle$? Write your final answer in $ax + by + cz = d$ form.

$$z - 2 = 1(x-4) - 3(y+1)$$

$$\Rightarrow -2 + 4 + 3 = x - 3y - z$$

$$\Rightarrow \boxed{x - 3y - z = 5}$$

8. (5 points) Find the rate of change of $f(x, y, z) = x\sqrt{y^2 + z^2}$ at the point $P = (2, 3, 4)$ in the direction of $\langle 1, 0, 5 \rangle$.

$$f_x = \sqrt{y^2 + z^2}$$

$$f_y = \frac{xy}{\sqrt{y^2+z^2}}$$

$$f_z = \frac{xz}{\sqrt{y^2+z^2}}$$

$$\hat{u} = \frac{\langle 1, 0, 5 \rangle}{\sqrt{26}}$$

$$= D_{\hat{u}} f(P) = \left\langle 5, \frac{6}{5}, \frac{8}{5} \right\rangle \cdot \frac{\langle 1, 0, 5 \rangle}{\sqrt{26}}$$

$$= \frac{5 + 8}{\sqrt{26}} = \boxed{\frac{13}{\sqrt{26}}} \text{ or } \boxed{\frac{\sqrt{26}}{2}}$$

T2, 4

9. (9 points) Let $f(x,y) = x\ln(x^2+y^2)$, $x(u,w) = 3uw$, $y = y(u,w)$, $y(-1,1) = 1$, and $y_u(-1,1) = -2$. Find $\frac{df}{du}$ when $u = -1$ and $w = 1$. Show organized work.

$x(-1,1) = -3$

$$f_u = f_x x_u + f_y y_u.$$

$$f_x = \left(\ln(x^2+y^2) + \frac{2x^2}{x^2+y^2}\right)\Bigg|_{(-3,1)} = \ln(10) + \frac{18}{10}$$

$$f_y = \frac{2xy}{x^2+y^2}\Bigg|_{(-3,1)} = \frac{-6}{10}. \qquad x_u(-1,1) = 3w\big|_{w=1} = 3.$$

$$\therefore f_u = \left(\ln(10) + \frac{18}{10}\right)\cdot 3 + \left(\frac{-6}{10}\right)\cdot(-2)$$

$$= \boxed{3\ln(10) + \frac{66}{10}} \text{ or } \boxed{3\ln(10) + 6.6} \text{ or } \boxed{3\ln(10) + \frac{33}{5}}.$$

10. (9 points) Find the moment of inertia about the origin, I_0, for the solid hemisphere $0 \leq z \leq \sqrt{1-x^2-y^2}$ if the density is $\delta(x,y,z) = 4\sqrt{x^2+y^2+z^2}$.

$$I_0 = \iiint_E \rho^2 \cdot \delta\, dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^2 \cdot 4\rho \cdot \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 4\int_0^{2\pi} d\theta \cdot \int_0^{\frac{\pi}{2}} \sin\phi\, d\phi \cdot \int_0^1 \rho^5\, d\rho$$

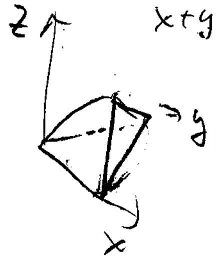
$$= 8\pi \cdot \left(-\cos\phi\Big|_0^{\frac{\pi}{2}}\right) \cdot \left(\frac{\rho^6}{6}\Big|_0^1\right)$$

$$= 8\pi(1)\left(\frac{1}{6}\right) = \boxed{\frac{4\pi}{3}}$$

T2, 5

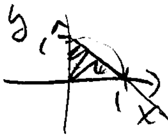
11. (7 points) Find the mass of the solid that has nonnegative x, y, and z coordinates for which $z^2 \le 4x$ and $x^2 + y^2 \le 1$ and if its density is $\delta(x, y, z) = \cancel{xy^2}$ $\cancel{2y^2}$ z

$x + y \le 1$



$$\text{Mass} = \int_0^1 \int_0^{1-x} \int_0^{2\sqrt{x}} z \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{z^2}{2} \Big|_0^{2\sqrt{x}} dy \, dx$$



$x+y=1$

$$= \int_0^1 \int_0^{1-x} 2x \, dy \, dx$$

$$= \int_0^1 2xy \Big|_0^{1-x} dx$$

$$= 2\int_0^1 x(1-x) \, dx$$

$$= 2\int_0^1 x - x^2 \, dx$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right)$$

$$= \boxed{\frac{1}{3}}$$

12. (7 points) Use Green's Theorem to find the area of the region R bounded by the closed curve C parameterized by $\vec{p}(t) = \langle \cos(t) - \sin(t), \sin(t) + \cos(t) \rangle$ for $0 \le t \le 2\pi$.

$$\text{Area} = \iint_R dA$$

$$\underset{G.T.}{=} \left| \int_0^{2\pi} \langle 0, \cos(t) - \sin(t) \rangle \cdot \langle -\sin(t) - \cos(t), \cos(t) - \sin(t) \rangle \, dt \right|$$

$$= \left| \int_0^{2\pi} \cos^2(t) - 2\sin(t)\cos(t) + \sin^2(t) \, dt \right|$$

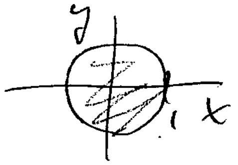
$$= \left| \int_0^{2\pi} 1 - \cancel{\sin(2t)}^{\to 0;\, 2\pi} \, dt \right|$$

$$= 2\pi$$

T2, 6

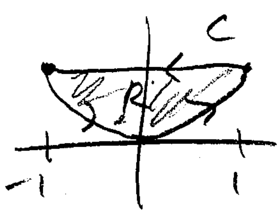
13. (6 points) Write $I = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{x^2+y^2} \frac{1}{\sqrt{x^2+y^2+z^2}}\,dz\,dy\,dx$ using cylindrical coordinates.

Do not evaluate or simplify!



$$\boxed{I = \int_0^{2\pi}\int_0^1\int_0^{r^2} \frac{1}{\sqrt{r^2+z^2}} \cdot r\,dz\,dr\,d\theta}$$

14. (7 points) Use Green's Theorem and a double integral to evaluate $I = \int_C y\,dx + x^2\,dy$ if C is the closed curve that follows the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$ and then returns on the straight line from $(1, 1)$ to $(-1, 1)$.



$$I \underset{G.T.}{=} \iint_R 2x-1\,dA$$

$$= \int_{-1}^{1}\int_{x^2}^{1} 2x-1\,dy\,dx$$

$$= \int_{-1}^{1} (2x-1)\left[1-x^2\right]dx$$

$$= \int_{-1}^{1} \underbrace{\cancel{2x-2x^3}}_{\text{ODD}}^{\to 0} \underbrace{-1+x^2}_{\text{even}}\,dx = 2\cdot\left[-x+\frac{x^3}{3}\right]_0^1 = \boxed{-\frac{4}{3}}$$