Test 3 100 points Math 200 Name: ______________________________

1. (9 points) Find the equation of the tangent plane to the surface parameterized by $\vec{p}(u, w) = \langle u^2 - w^2, 2u + w, \cancel{4}$ $4\sqrt{w}\rangle$ at the point where $u = 1$ and $w = 4$.

$$\vec{p}_u = \langle 2u, 2, 0\rangle\Big|_{(1,4)} = \langle 2, 2, 0\rangle$$

$$\times\ \vec{p}_v = \langle -2w, 1, 2w^{-\frac{1}{2}}\rangle\Big|_{(1,4)} = \langle -8, 1, 1\rangle$$

$$\langle 2, -2, 18\rangle$$

Use $\vec{n} = \langle 1, -1, 9\rangle$.

$$\vec{p}(1,4) = \langle 1-16, 2+4, 4\sqrt{4}\rangle$$

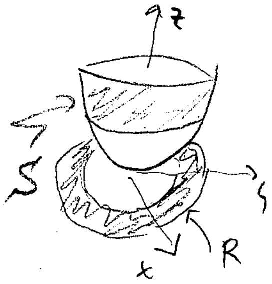
$$= \langle -15, 6, 8\rangle$$

$$x - y + 9z = \langle 1, -1, 9\rangle \cdot \langle -15, 6, 8\rangle$$

$$\Rightarrow x - y + 9z = -15 - 6 + 72$$

$$\Rightarrow \boxed{x - y + 9z = 51}$$

2. (8 points) Find the mass of the piece of the ~~parabola~~ paraboloid $z = x^2 + y^2$ that lies between $z = 1$ and $z = 2$ if density is $\delta(x, y, z) = \dfrac{z - x}{\sqrt{4x^2 + 4y^2 + 1}}$ g/cm^2.



$$d\vec{S} = \pm\langle -2x, -2y, 1\rangle\,dx\,dy$$

$$dS = \sqrt{4x^2 + 4y^2 + 1}\,dx\,dy$$

$$\text{Mass} = \iint_S \delta\,dS$$

$$= \iint_R \frac{(x^2+y^2) - x}{\cancel{\sqrt{4x^2+4y^2+1}}}\cancel{\sqrt{4x^2+4y^2+1}}\,dx\,dy$$

$$= \int_0^{2\pi}\int_1^{\sqrt{2}} (r^2 - r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{r^4}{4} - \frac{r^3}{3}\cos(\theta)\right)\Bigg|_1^{\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} \left(1 - \cancelto{0}{\frac{\sqrt{8}}{3}\cos(\theta)}\right) - \left(\frac{1}{4} - \cancelto{0}{\frac{1}{3}\cos(\theta)}\right) d\theta = 2\pi \cdot \frac{3}{4} = \boxed{\frac{3\pi}{2}}$$

T3, 2

③ (9 points) Use a surface integral to find the flux of $\vec{F}(x,y,z) = \langle \overset{X}{\cancel{1}}, x, xyz\rangle$ **out of** the piece of the cylinder $x^2+y^2=1$ that is above the plane $z=0$ and below the plane $z=x$.

$$d\vec{S} = \frac{\langle x,y,0\rangle}{1} \cdot 1\, d\theta\, dz$$

$$\text{Flux} = \int_0^{2\pi}\int_0^{1\cdot\cos\theta} \langle X, x, xyz\rangle \cdot \langle x, y, 0\rangle\, dz\, d\theta \quad ; \quad x^2+xy = \cos^2\theta + \cos\theta\sin\theta$$

$$= \int_0^{2\pi} (\cos^2\theta + \cos\theta\sin\theta)\cdot\left(z\Big|_0^{\cos\theta}\right.$$

$$= \int_0^{2\pi} \cos^3\theta + \cos^2\theta\sin\theta\, d\theta = \int_0^{2\pi} \cos\theta(1-\sin^2\theta)\,d\theta - \left(\frac{\cos^3\theta}{3}\Big|_0^{2\pi}\right.$$

$$= \int_0^{2\pi} \overset{\nearrow 0;\, 1T}{\cancel{\cos\theta}}\, d\theta - \int_0^{2\pi} \sin^2\theta\cos\theta\, d\theta - 0 = -\frac{\sin^3\theta}{3}\Big|_0^{2\pi} = \boxed{0}$$

4. (8 points) Find and classify all critical points for $f(x,y) = x^3 - 6xy + y^3$. A complete answer will include either the word "local" or "global."

① $f_x = 3x^2 - 6y = 0 \Rightarrow y = \frac{x^2}{2}$

② $f_y = -6x + 3y^2 = 0 \Rightarrow 3y^2 = 6x$.

Notice $\boxed{(0,0) \text{ is a c.p.}}$ and $x \neq 0 \Rightarrow y \neq 0$, So $x \neq 0 \Rightarrow \frac{3y^2}{5} = \frac{6x}{\left(\frac{x^2}{2}\right)} \Rightarrow \underline{\underline{y}} = \frac{4}{x}$.

Subustituting into ① $\Rightarrow \frac{4}{x} = \frac{x^2}{2} \Rightarrow 8 = x^3 \Rightarrow x=2 \Rightarrow y=2$

So $\boxed{(2,2) \text{ is the other c.p.}}$

$$\left.\begin{aligned} f_{xx} &= 6x \\ f_{xy} &= -6 \\ f_{yy} &= 6y \end{aligned}\right\} \Rightarrow D(x,y) = \begin{vmatrix} 6x & -6 \\ -6 & 6y \end{vmatrix}.$$

$D(0,0) = -36 < 0 \Rightarrow \boxed{(0,0) \text{ is a saddle}}$

$D(2,2) = \begin{vmatrix} 12 & -6 \\ -6 & 12 \end{vmatrix} = 144 - 36 > 0$, $f_{xx}(2,2) = 12 > 0$

$\Rightarrow \boxed{(2,2) \text{ is a local minimum.}}$

"local" b/c $\lim_{y\to-\infty} f(0,y) = \lim_{y\to-\infty} y^3 = -\infty$.

T3,3

5. (8 points) Evaluate $I = \iint_R \frac{x-2y}{3x-y}\, dA$ using a substitution if R is bounded by $x-2y=0$, $x-2y=2$, $3x-y=1$, and $3x-y=8$.

$u = x-2y$

$0 \le u \le 2$

$w = 3x-y$

$1 \le w \le 8$

$$\frac{\partial(u,w)}{\partial(x,y)} = \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}$$

$= -1+6 = 5$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{5}.$$

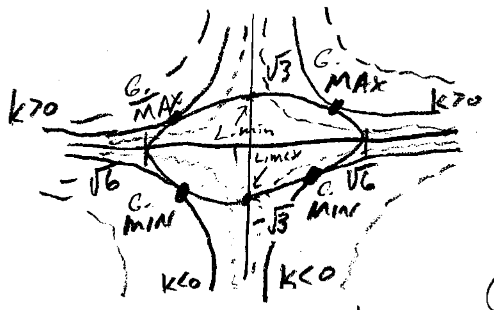
$$I = \int_0^2 \int_1^8 \frac{u}{w} \cdot \frac{1}{5}\, dw\, du$$

$$= \frac{1}{5} \int_0^2 u\, du \cdot \int_1^8 \frac{1}{w}\, dw$$

$$= \frac{1}{5} \left(\frac{u^2}{2}\Big|_0^2\right) \cdot \left(\ln(w)\Big|_1^8\right)$$

$$= \frac{1}{5} \cdot 2 \cdot \ln(8) = \boxed{\frac{2}{5}\ln(8)}$$

6. (9 points) Use Lagrange Multipliers to find the minimum value of $f(x,y) = x^2y$ if the inputs are constrained to lie on the ellipse $x^2+2y^2=6$. Sketch a contour plot with the significant level curves to verify your answer.



$K = x^2y \Rightarrow y = \frac{K}{x^2}$

$C(x,y) = x^2+2y^2$

$f(x,y) = x^2y$

$\nabla f = \lambda \nabla C$

$$\left.\begin{aligned} &① \ 2xy = 2\lambda x \\ &② \ x^2 = 4\lambda y \\ &③ \ x^2+2y^2 = 6 \end{aligned}\right\}$$

$x=0$ and ② $\Rightarrow y=0$, but $(0,0)$ is not on the ellipse.

$\therefore$ assume $x \ne 0 \ne y \Rightarrow \lambda \ne 0$.

Then $\frac{①}{②} \Rightarrow \frac{2y}{x} = \frac{x}{2y} \Rightarrow x^2 = 4y^2$.

③ $\Rightarrow 6y^2 = 6 \Rightarrow y = \pm 1 \Rightarrow x = \pm 2$.

Picture $\Rightarrow$ Max $= f(2,1) = 4$ (and $f(-2,1)$)

and

$\boxed{\text{Min} = f(2,-1) = -4}$.

and $f(-2,-1)$

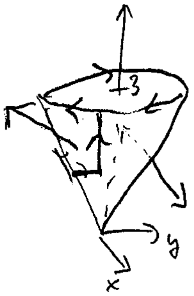
T3, 4

*⑦ (8 points) Use Stokes' Theorem **once** to find $I = \iint\limits_S \nabla \times \vec{F} \cdot d\vec{S}$ if $\vec{F}(x,y,z) = \langle y, xyz, y^2z^2 \rangle$ and S is the piece of the cone $z^2 = x^2 + y^2$ between $z = 0$ and $z = 3$ oriented ~~up~~ down.
Hint: you must evaluate a line integral.

$$I \underset{S.T.}{=} \oint_{Bd(S)} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \langle -3\sin(t), -27\cos(t)\sin(t), -81\sin^2(t) \rangle \cdot \langle -3\sin(t), -3\cos(t), 0 \rangle d$$

$$Bd(S):\ \vec{\alpha}(t) = \langle \overset{x}{3\cos(t)}, \overset{y}{-3\sin(t)}, \overset{z}{3} \rangle$$

$$0 \le t \le 2\pi$$



$$\vec{\alpha}'(t) = \langle -3\sin(t), -3\cos t, 0 \rangle$$

$$= \int_0^{2\pi} 9\sin^2(t) + 81\cos^2(t)\sin(t)\ dt$$

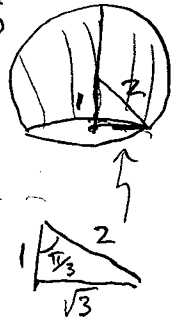
$$= \int_0^{2\pi} \frac{9}{2}(1 - \overset{0;\,2\pi}{\cancel{\cos(2t)}})dt - 81 \left. \frac{\cos^3(t)}{3} \right|_0^{2\pi}$$

$$= \frac{9}{2} \cdot 2\pi - 0 = \boxed{9\pi}$$

8. (9 points) Use a surface integral to find the flu~~x of $\vec{F}(x,y,z) = \langle y, -x, z \rangle$ through~~ the piece of the sphere $x^2 + y^2 + z^2 = 4$ that lies above $z = -1$ if ~~the~~ it is oriented ~~up~~. out

S

sphere



$$\pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

$$\therefore 0 \le \phi \le \frac{2\pi}{3}$$

$$0 \le \theta \le 2\pi$$

$$d\vec{S} = \frac{\langle x, y, z \rangle}{2} \cdot 4\sin\phi\, d\phi\, d\theta$$

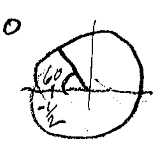
$$\text{Flux} = \iint\limits_S \vec{F} \cdot d\vec{S}$$

$$= \int_0^{2\pi}\int_0^{\frac{2\pi}{3}} \langle y, -x, z \rangle \cdot \frac{\langle x, y, z \rangle}{2} \cdot 4\sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{2\pi}{3}} 2z^2 \sin\phi\, d\phi\, d\theta \quad ; \quad z = \rho\cos\phi = 2\cos\phi.$$

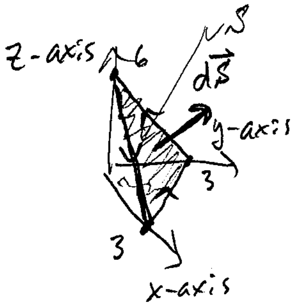
$$= 4\pi \int_0^{2\pi/3} 4\cos^2\phi \sin\phi\, d\phi$$

$$= 16\pi \left(-\frac{\cos^3\phi}{3} \right|_0^{2\pi/3} = \frac{16\pi}{3}\left(\underbrace{-\left(\frac{-1}{2}\right)^3 + 1}_{=\frac{9}{8}} \right) = \boxed{6\pi}$$



T3,5

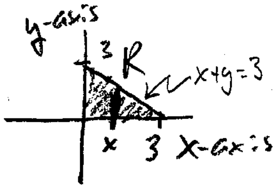
9. (9 points) Use Stokes Theorem to find the work done by $\vec{F}(x,y,z) = \langle z, \overset{2x}{x}, \overset{4y}{\cancel{y}} \rangle$ on a particle that moves around the triangle from vertex $(3,0,0)$ to vertex $(0,3,0)$ to vertex $(0,0,6)$ before returning to $(3,0,0)$. Hint: You must use a double integral.



S: $2x+2y+z=6$

$\boxed{z = 6-2x-2y}$

$\Rightarrow d\vec{S} = \langle 2,2,1\rangle\,dx\,dy$



$$\text{Work} = \oint_{Bd(S)} \vec{F}\cdot d\vec{s} \underset{\uparrow\, S.T.}{=} \iint_{S} (\nabla \times \vec{F}) \cdot \langle 2,2,1\rangle\,dx\,dy$$

$$= \iint_R \langle 4, 1, 2\rangle \cdot \langle 2,2,1\rangle\,dx\,dy$$

$$= (8+2+2)\cdot \text{area}(R)$$

$$= 12 \cdot \tfrac{1}{2}\cdot 3 \cdot 3$$

$$= \boxed{54}$$

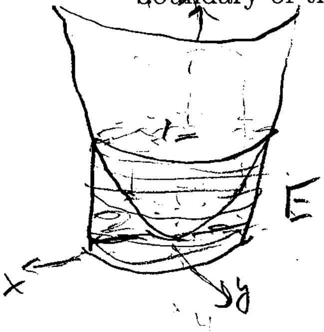
10. (9 points) Use the Divergence Theorem **once** to find the flux of $\vec{F}(x,y,z) = \langle \overset{2xy^2}{2xy}, z^2, -2y\rangle$ out of the boundary of the solid ~~half cylinder $x^2+y^2 \le 1$ for $y \ge 0$ that lies between $z = 0$ and $z = x^2+y^2$.~~

Keep Bold

bounded by the

Hint: you must use a triple integral.

Under $z = x^2+y^2$, above $z=0$, and inside



intersection

$$\left.\begin{matrix} x^2+y^2=1 \\ z = x^2+y^2 \end{matrix}\right\} \Rightarrow z = 1.$$

$$\text{Flux} = \iint_{Bd(E)} \vec{F}\cdot d\vec{S}$$

$$\underset{D.T.}{=} \iiint_E \nabla\cdot\vec{F}\,dV$$

$$= \iiint_E 2y^2 + 0 - 0\,dV$$

$$\underset{\text{cyl coords}}{=} \int_0^{2\pi}\int_0^1\int_0^{r^2} 2(r^2\sin^2\theta)\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 2r^5\sin^2\theta\,dr\,d\theta$$

$$\int_0^{2\pi} 1 - \overset{0;\,2\pi}{\cancel{\cos(2\theta)}}\,d\theta \cdot \left(\frac{r^6}{6}\right)\Big|_0^1 = 2\pi\cdot\frac{1}{6} = \boxed{\frac{\pi}{3}}$$

T3,6

11. (8 points) Use the Divergence Theorem to find $I = \iint\limits_{Bd(E)} x^2 + 2y^2 - 4z\, dS$ if E is the solid ball $x^2 + y^2 + z^2 \leq 4$.

$$I = \iint\limits_{B\partial(E)} 2\cdot\langle x, 2y, -4\rangle \cdot \overbrace{\frac{\langle x, y, z\rangle}{2}\, dS}^{d\vec{S}}$$

$$\overset{}{\underset{D.T.}{=}} \iiint\limits_{E} 2\left[1 + 2 + 0\right] dV$$

$$= 6 \text{ vol}(E)$$

$$= \cancel{6}^{2} \cdot \frac{4}{\cancel{3}}\pi(2)^3$$

$$= \boxed{64\pi}$$

12. (6 points) The velocity field of a fluid $\vec{v}$ $\overset{g\cdot}{\text{cm}}$/sec has a divergence at the point $P = (2, 2, 2)$ of $\overset{g}{}$3/~~sec?~~. Write $3\,{}^{?}/\text{sec}$?? just as you did in HW #8 ~~theep~~
Estimate the flow rate (or flux) out of the sphere of radius 0.1 cm centered at P.

$$\nabla\cdot\vec{v}(P)\,{}^{g}/_{sec} \approx \frac{\oiint\limits_{\rho=0.1} \vec{v}\cdot d\vec{S}\ \ \frac{g\cdot cm^3}{sec}}{\text{Vol}(\rho \leq 0.1)\ \ cm^3}$$

$$\Rightarrow 3 \approx \frac{\oiint \vec{v}\cdot d\vec{S}}{\frac{4}{3}\pi(0.1)^3}$$

$$\Rightarrow 4\pi(0.1)^3 = \oiint \vec{v}\cdot d\vec{S}$$

$$\Rightarrow \boxed{\frac{4\pi}{1000} = \frac{\pi}{250} = \oiint\limits_{\rho=0.1} \vec{v}\cdot d\vec{S}}$$