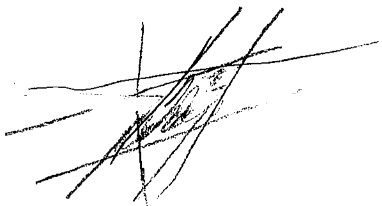
Quiz 7 25 points Math 200 Name: ______________________

1. (7 points) R is the region inside the parallelogram in the xy - plane bounded by $x - 2y = \cancel{0}\ 1$, $x - 2y = \cancel{4}\ e$, $3x - y = \cancel{1}\ 0$, and $3x - y = 8$. Change variables to find $I = \iint\limits_R \frac{3x - y}{x - 2y}\, dx\, dy$.



$$I = \int_1^e \int_0^8 \frac{v}{u} \cdot \frac{1}{5}\, dv\, du$$

$1 \le u = x - 2y \le e$

$0 \le v = 3x - y \le 8$

$$\frac{\partial(u,v)}{\partial(x,y)} = \left| \begin{matrix} 1 & -2 \\ 3 & -1 \end{matrix} \right| = 5$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{5}.$$

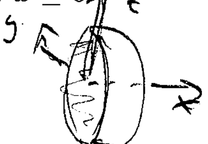
$$= \frac{1}{5} \int_1^e \frac{1}{u}\, du \cdot \int_0^8 v\, dv$$

$$= \frac{1}{5} (\ln(e) - 0) \left(\frac{v^2}{2} \Big|_0^8 \right)$$

$$= \frac{1}{5} \cdot 32$$

$$= \boxed{\frac{32}{5}}$$

2. (6 points) Find the flux of $\vec{F}(x, y, z) = \langle x, z, -y \rangle$ through the hemisphere $x^2 + y^2 + z^2 = 9$ for $x \ge 0$. Oriented w/ positive x-coordinate



$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S} = \int_0^{\pi} \int_{-\pi/2}^{\pi/2} \langle x, z, -y \rangle \cdot \frac{\langle x, y, z \rangle}{3} \cdot 3^2 \sin\phi\, d\theta\, d\phi$$

$$= 3 \int_0^{\pi} \int_{-\pi/2}^{\pi/2} x^2 \sin\phi\, d\theta\, d\phi$$

$$= 3 \int_0^{\pi} \int_{-\pi/2}^{\pi/2} 9 \sin^2\phi \cos^2\theta \sin\phi\, d\theta\, d\phi$$

$$= 27 \int_{-\pi/2}^{\pi/2} \frac{1 + \cancel{\cos(2\theta)}^{\,0;\,\pi}}{2}\, d\theta \cdot \int_0^{\pi} \sin\phi - \sin\phi \cos^2\phi\, d\phi$$

$$= 27 \cdot \frac{\pi}{2} \cdot \left(-\cos\phi + \frac{\cos^3\phi}{3} \Big|_0^{\pi} \right) = \frac{27\pi}{2} \left(\frac{2}{3} + \frac{2}{3} \right)$$

$$= \boxed{18\pi}$$

OR (x replaces z) $x = 3\cos\phi$

$$3 \int_0^{2\pi} \int_0^{\pi/2} x^2 \sin\phi\, d\phi\, d\theta$$

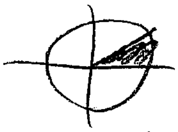
$$= 3 \int_0^{2\pi} \int_0^{\pi/2} 9 \cos^2\phi \sin\phi\, d\phi\, d\theta$$

$$= 9 \cdot 2\pi \left(-3 \right) \frac{\cos^3\phi}{3} \Big|_0^{\pi/2}$$

$$= 18\pi\, [1 - 0]$$

$$= 18\pi$$

3. (6 points) Find $I = \iint\limits_S \delta(x,y,z)\, dS$ if $\delta(x,y,z) = \sqrt{x^2+y^2+z^2}$ and if S is the piece of the cone $z = \sqrt{x^2+y^2}$ for which $0 \le x^2+y^2 \le 1$ and $0 \le y \le x$.



$$d\vec{S} = \pm\left\langle \frac{-x}{z}, \frac{-y}{z}, 1\right\rangle dx\,dy$$

$$\Rightarrow dS = \sqrt{\frac{x^2+y^2}{z^2}+1}\, dx\,dy = \sqrt{2}\, dx\,dy \quad (\text{since } z^2 = x^2+y^2)$$

Also $\delta = \sqrt{2(x^2+y^2)}$, so

$$I = \int_0^{\pi/4}\int_0^1 \sqrt{2}\, r \cdot \sqrt{2} \cdot r\, dr\, d\theta$$

$$= 2 \cdot \frac{\pi}{4} \cdot \left(\frac{r^3}{3}\Big|_0^1\right) = \boxed{\frac{\pi}{6}.}$$

4. (6 points) Find the mass of the piece of the cylinder $\cancel{r = 2}$ $x^2+y^2=4$ that lies above $z = 0$ and below the saddle $z = 4 + x^2 - y^2$ if the density is $\delta(x,y,z) = z$ grams per square centimeter. Hint: the relevant intersection is $z = 2x^2$.

$$\text{Mass} = \iint\limits_S \delta\, dS.$$

S: $r = 2$ below $z = 4+x^2-y^2$.

intersection: $\begin{cases} x^2+y^2=4 \\ z = 4+x^2-y^2 \end{cases}$

$$\Rightarrow z = x^2+y^2+x^2-y^2$$

$$\Rightarrow z = 2x^2 = 2r^2\cos^2\theta$$

$$r = 2 \Rightarrow z = 8\cos^2\theta.$$

$$dS = 2 \cdot dz\, d\theta.$$

$$\therefore \text{Mass} = \int_0^{2\pi}\int_0^{8\cos^2\theta} z \cdot 2\, dz\, d\theta$$

$$= \int_0^{2\pi} z^2\Big|_0^{8\cos^2\theta} d\theta$$

$$= \int_0^{2\pi} 64\cos^4\theta\, d\theta$$

$$= 16\int_0^{2\pi} (1+\cos(2\theta))^2\, d\theta$$

$$= 16\int_0^{2\pi} 1 + \cancel{2\cos(2\theta)} + \cos^2(2\theta)\, d\theta \quad 0;\ 2\pi$$

$$= 16\int_0^{2\pi} \frac{3}{2} + \cancel{\frac{\cos(4\theta)}{2}}\, d\theta = \boxed{48\pi} \quad 0;\ 4\pi$$