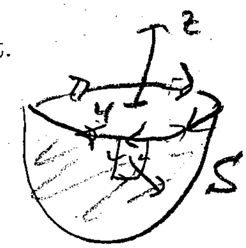
Quiz 8 25 points Math 200 Name: ______________________

1. (7 points) Use Stokes' Theorem exactly once to find $I = \iint_S \nabla \times \vec{F} \cdot d\vec{S}$ if $\vec{F}(x,y,z) = \langle z, -x, zy \rangle$ and if S is the piece of $z = x^2 + y^2$ for $0 \le z \le 4$ oriented with negative z component.



$$I \underset{S.T.}{=} \int_{B\partial(S)} \vec{F} \cdot d\vec{s}$$

$\therefore B\partial(S) = x^2+y^2=4$, $z=4$ C.W.

$\vec{\alpha}(t) = \langle 2\cos(t), -2\sin(t), 4 \rangle$, $0 \le t \le 2\pi$

$$= \int_0^{2\pi} \langle 4, -2\cos(t), -4\cdot 2\sin(t) \rangle \circ \langle -2\sin(t), -2\cos(t), 0 \rangle \, dt$$

$$= \int_0^{2\pi} -8\sin(t) + 4\cos^2(t) + 0 \, dt \quad \text{(0; 1T)}$$

$$= 2\int_0^{2\pi} 1 + \cos(2t) \, dt = \boxed{4\pi} \quad \text{(0; 2T)}$$

2. (6 points) Use the divergence theorem exactly once to find the mass of the surface S equal to $x^2 + y^2 + z^2 = 4$ if the density is $\delta(x,y,z) = 3x^2 + xy + yz$ grams/cm^2.

$$\text{MASS} = \iint_S \delta \, dS \; ; \; \delta \, dS = (3x^2 + xy + yz) \cdot 2^2 \sin\phi \, d\phi \, d\theta$$

$$= \langle 3x, x, y \rangle \cdot \underbrace{\langle x, y, z \rangle \cdot 2 \sin\phi \, d\phi \, d\theta}_{= 2 d\vec{S}}$$

$$= 2\iint_S \langle 3x, x, y \rangle \cdot d\vec{S}$$

$$\underset{D.T.}{=} 2\iiint_{\rho \le 2} 3 + 0 + 0 \, dV$$

$$= 6 \cdot \text{Vol}(\rho \le 2) = 6 \cdot \frac{4}{3}\pi \cdot 2^3 = \boxed{64\pi}$$

3. (6 points) Use the Divergence Theorem to find the flux of $\vec{F}(x,y,z) = (2xy)\hat{\imath} + (3y^2)\hat{\jmath} - (2zy)\hat{k}$ out of the boundary of the solid cube, T, spanned by the vectors $2\hat{\imath}$, $2\hat{\jmath}$, and $2\hat{k}$ with tails at the origin.

$$\text{Flux} = \iint_{B\partial(T)} \vec{F}\cdot d\vec{S}$$

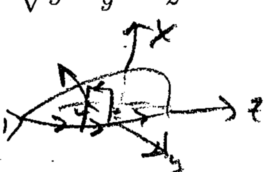
$$\overset{D.T.}{=} \iiint_T 2y + 6y - 2y \, dV$$

$$= \int_0^2\int_0^2\int_0^2 6y \, dy\, dx\, dz$$

$$= 2\cdot 2\cdot \left(3y^2\Big|_0^2\right)$$

$$= \boxed{48}$$

4. (6 points) Verify Stokes theorem if $\vec{F}(x,y,z) = \langle -xz, x, \frac{y^2}{2}\rangle$ and the surface is $x = \sqrt{9-y^2-z^2}$ oriented away from the origin.
Hint: evaluate both a line integral and a surface integral and show they are equal.



$$\vec{P}(y,z) = \langle \sqrt{9-y^2-z^2}, y, z\rangle$$

$$\Rightarrow d\vec{S} = \oplus\langle 1, \frac{y}{x}, \frac{z}{x}\rangle dy\, dz$$

(OR) use spherical parameterization $\int_{-\pi/2}^{\pi/2}\int_0^{\pi} \cdots \, d\phi\, d\theta$

$$\iint_S \nabla\times\vec{F}\cdot d\vec{S}$$

$$= \iint_{y^2+z^2\le 9} \langle y, -x, 1\rangle\cdot\langle 1, \frac{y}{x}, \frac{z}{x}\rangle\, dy\, dz$$

(yz-plane polar) $$\int_0^{2\pi}\int_0^3 \left(y - y + \frac{z}{x}\right)\cdot r\, dr\, d\theta \; ; \; x = \sqrt{9-r^2}, \; z = 3\sin\theta$$

$$= \int_0^{2\pi} 3\sin\theta\, d\theta \cdot \int_0^3 \frac{r}{\sqrt{9-r^2}}\, dr \qquad (\to 0;\, 1T)$$

$$= 0\cdot\left(-(9-r^2)^{\frac{1}{2}}\Big|_0^3\right) = \boxed{0}$$

$B\partial(S)$: $\vec{\alpha}(t) = 3\langle 0, \cos(t), \sin(t)\rangle$, $0\le t\le 2\pi$

$$\int_{B\partial(S)} \vec{F}\cdot d\vec{s}$$

$$= \int_0^{2\pi} \langle 0, 0, \frac{9\cos^2(t)}{2}\rangle \cdot 3\langle 0, -\sin(t), \cos(t)\rangle dt$$

$$= \frac{27}{2}\int_0^{2\pi} \cos^3(t)\, dt \; ; \; \cos^3 t = \cos t\,(1-\sin^2 t)$$

$$= \frac{27}{2}\int_0^{2\pi} \cos(t) - \cos(t)\sin^2(t)\, dt \qquad (\to 0;\, 1T)$$

$$= -\frac{27}{2}\left(\frac{\sin^3(t)}{3}\right)\Big|_0^{2\pi}$$

$$= \boxed{0}$$

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