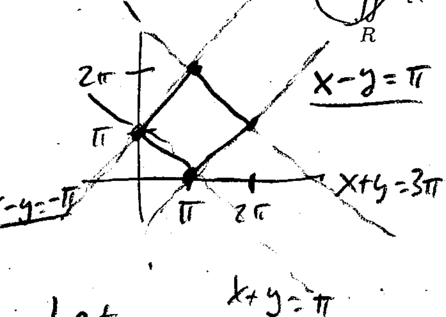
Quiz 7 25 points Math 200 Name: ____________________

1. (~~5~~ 7 points) R is the square in the xy - plane with vertices $(\pi, 0)$, $(\pi, 2\pi)$, $(0, \pi)$, and $(2\pi, \pi)$. Change variables to find $I = \iint_R [(x-y)\sin(x+y)]^2 \, dx\, dy$.



Let
$-\pi \le u = x - y \le \pi$
$\pi \le v = x + y \le 3\pi$

$$\frac{\partial(u,v)}{\partial(x,y)} = \left|\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}\right| = 2$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}.$$

$$\therefore I = \int_{\pi}^{3\pi}\int_{-\pi}^{\pi} u^2 \cdot \sin^2(v) \cdot \frac{1}{2}\, du\, dv$$

even

$$\Rightarrow I = \int_0^{\pi} u^2 du \cdot \int_{\pi}^{3\pi} \frac{1-\cos(2v)}{2} dv$$

$$\Rightarrow I = \frac{u^3}{3}\Big|_0^{\pi} \cdot \frac{1}{2}(2\pi)$$

$$\Rightarrow \boxed{I = \frac{\pi^4}{3}}$$

2. (~~5~~ 6 points) Find the mass of the piece of the cylinder $x^2 + y^2 = 1$ that lies in the first octant ($x \ge 0$, $y \ge 0$, $z \ge 0$) and below $z = 1 - x$ if the density is $\delta(x, y, z) = 1 + y$ grams per square centimeter.

$dS = 1 \cdot dz\, d\theta$

$x = 1\cdot\cos\theta$, $y = 1 \cdot \sin\theta$

$$\Rightarrow \text{Mass} = \int_0^{\pi/2}\int_0^{1-\cos\theta} (1+\sin\theta)\cdot dz\, d\theta$$

$$= \int_0^{\pi/2} (1+\sin\theta)(1-\cos\theta)\, d\theta$$

$$= \int_0^{\pi/2} 1 + \sin\theta - \cos\theta - \sin\theta\cos\theta\, d\theta$$

$$= \frac{\pi}{2} + \left(\cos\theta + \sin\theta - \frac{\sin^2\theta}{2}\right)\Big|_0^{\pi/2} = \frac{\pi}{2} + \left(0 + 1 - \frac{1}{2} - 1\right) = \boxed{\frac{\pi - 1}{2}}$$ grams

3. (~~5~~ points) Find the flux of $\vec{F}(x, y, z) = \langle 1 - x, y - x + 1, z \rangle$ through the surface parameterized by $\vec{p}(u, v) = \langle 1 - u, v - u, uv \rangle$, oriented up, if $0 \le u \le 1$ and $0 \le v \le$ ~~0.2~~ 2

DELETE

$\vec{p}_u du = \langle -1, -1, v\rangle du$

$+ \ \vec{p}_v dv = \langle 0, 1, u \rangle dv$

$d\vec{S} = \pm\langle -u-v, u, -1\rangle du\, dv$

oriented up

$\Rightarrow d\vec{S} = \langle u+v, -u, 1\rangle du\, dv$

$$\Phi = \int_0^1\int_0^2 \langle 1-(1-u), (v-u)-(1-u)+1, uv\rangle \cdot d\vec{S}$$

$$= \int_0^1\int_0^2 u^2 + uv - uv + uv\, dv\, du$$

$$= \int_0^1\int_0^2 u^2 + uv\, dv\, du$$

$$= \int_0^1 6u^2 + \left(\frac{uv^2}{2}\Big|_0^2\right) du = \int_0^1 6u^2 + 2u\, du$$

$$= 2u^3 + u^2\Big|_0^1 = \boxed{3}$$

③ ~~4.~~ (~~5~~ 6 points) Find $I = \iint_S \vec{F} \cdot d\vec{S}$ if $\vec{F}(x,y,z) = < y, x, -z >$ if S is the piece of $x+y+z=4$, oriented up, that lies inside the cylinder $x^2+y^2=1$. Hint: the surface is a plane, not a cylinder.

$z = 4 - x - y = f(x,y)$

$\Rightarrow d\vec{S} = <1,1,1> dx\,dy$

oriented up

$I = \iint_{r\le 1} <y, x, x+y-4> \cdot <1,1,1>\, dx\,dy$

$I = \iint_{r\le 1} y + x + x + y - 4 \, dx\,dy$

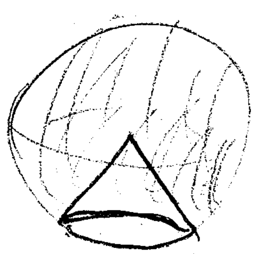
$I = \int_0^{2\pi}\int_0^1 2r(\cos\theta + \sin\theta) \cdot r\,dr\,d\theta - 4\cdot\pi$

$= \left.\frac{2r^3}{3}\right|_0^1 \cdot \int_0^{2\pi} \cancel{\cos\theta + \sin\theta}\, d\theta - 4\pi$

$= \frac{2}{3}\cdot 0 - 4\pi$

$= -4\pi$

④ ~~5.~~ (~~5~~ 6 points) Find the mass of the piece of the sphere $\rho = 2$ that lies above the cone $\phi = \cancel{\frac{3\pi}{4}}\ \frac{5\pi}{6}$ if the density is $\delta(x,y,z) = x^2 + y^2$ grams per square centimeter.



$\rho = 2$

$\text{Mass} = \iint_S \delta\, dS$

$dS = 4\sin\phi\, d\phi\, d\theta$

$\delta = r^2 = \rho^2 \sin^2\phi = 4\sin^2\phi$

$\text{Mass} = \int_0^{2\pi}\int_0^{\frac{5\pi}{6}} 4\sin^2\phi \cdot 4\sin\phi\, d\phi\, d\theta$

$\Rightarrow \text{Mass} = 16 \cdot 2\pi \cdot \int_0^{\frac{5\pi}{6}} \sin\phi\,(1-\cos^2\phi)\, d\phi$

$= 32\pi \cdot \left.\left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\right|_0^{\frac{5\pi}{6}}$

$= 32\pi\left(\frac{\sqrt{3}}{2} - \frac{3\sqrt{3}}{24} + 1 - \frac{1}{3}\right)$

$= \cancel{32}^{4}\pi\left(\frac{9\sqrt{3}+16}{\cancel{24}_{3}}\right)$

$= \frac{4\pi}{3}(9\sqrt{3}+16)$