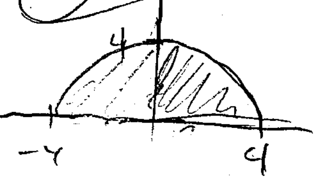
Math 200 Quiz #6 25 points Name:

1) Use polar coordinates to evaluate $I = \int_{\cancel{0}\ -4}^{4}\int_0^{\sqrt{16-x^2}} \tan^{-1}\left(\frac{y}{x}\right) dy\,dx$. (4 points)



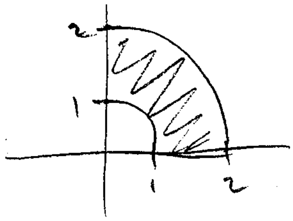
$$I = \int_0^{\pi}\int_0^{4} \cancel{\theta}\ r\,dr\,d\theta$$

$$= \int_0^{\pi} \theta\, d\theta \cdot \int_0^4 r\,dr$$

$$= \frac{\theta^2}{2}\Big|_0^{\pi} \cdot \left(\frac{r^2}{2}\Big|_0^4\right)$$

$$\therefore\quad I = \frac{\pi^2}{2}\cdot 8 = \boxed{4\pi^2}$$

2) Find $\bar{y}$, the y-coordinate of the center of mass of the lamina that is bounded by $1 \le x^2 + y^2 \le 4$ in the first quadrant (x and y are both nonnegative.) The density of the lamina is $\delta(x, y) = x$ grams per cm^2. (5 points)



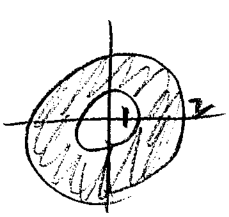
$$\text{Mass} = \int_0^{\pi/2}\int_1^2 r\cos\theta \cdot r\,dr\,d\theta = \int_0^{\pi/2}\cos\theta\,d\theta \cdot \int_1^2 r^2\,dr$$

$$= \sin\theta\Big|_0^{\pi/2} \cdot \left(\frac{r^3}{3}\Big|_1^2\right) = \frac{7}{3}$$

$$M_x = \int_0^{\pi/2}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta = \frac{\sin^2\theta}{2}\Big|_0^{\pi/2} \cdot \left(\frac{r^4}{4}\Big|_1^2\right) = \frac{15}{8}$$

$$\therefore\quad \bar{y} = \frac{15}{8}\cdot\frac{3}{7} = \boxed{\frac{45}{56}}$$

3) Find I_0, the moment of inertia about (0, 0), for the lamina that is ~~bounded by~~ $\cancel{1 \le}\ x^2 + y^2 \le 4$ ~~in the first quadrant (x and y are both nonnegative.)~~ if The density of the lamina is $\delta(x, y) = x^2$ grams per cm^2.
(4 points)



$$I_0 = \int_0^{2\pi}\int_1^2 r^2 \cdot r^2\cos^2\theta \cdot r\,dr\,d\theta$$

0; 2T

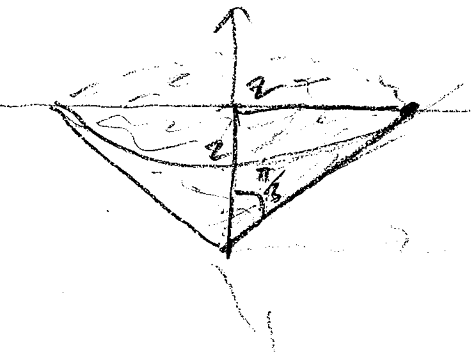
$$= \int_1^2 r^5\,dr \cdot \int_0^{2\pi} \frac{1+\cos(2\theta)}{2}\,d\theta$$

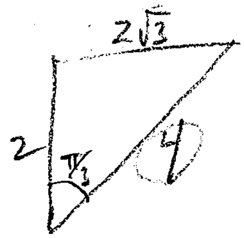
$I = \frac{1}{2}mv^2$
$= \frac{1}{2}mR^2\omega^2$
$(v = R\omega)$

$$= \frac{r^6}{6}\Big|_1^2 \cdot \pi = \frac{63}{6}\pi = \boxed{\frac{21}{2}\pi}$$

Note: In Cylindrical coordinates the integral would be: $\int_0^{2\pi}\int_0^{2\sqrt{3}}\int_{r/\sqrt{3}}^{2} z\,r\,dz\,dr\,d\theta$

4) Use spherical coordinates to evaluate $I = \iiint_E z\,dV$ if E is the solid that lies below the plane $z = 2$ and ~~simultaneously lies~~ above the cone $\phi = \cancel{\frac{\pi}{6}}$ $\frac{\pi}{3}$. (6 points)





$z = \rho\cos\phi$

$z = 2 \Rightarrow 2 = \rho\cos\phi$
$\Rightarrow \rho = 2\sec\phi$

$$I = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{2\sec\phi} \rho\cos\phi \cdot \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

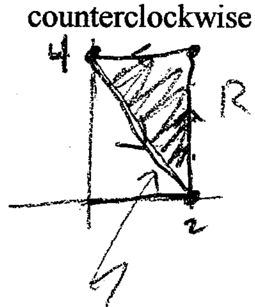
$$= 2\pi \int_0^{\pi/3} \cos\phi\sin\phi \cdot \frac{\rho^4}{4}\Bigg|_0^{2\sec\phi} d\phi$$

$$= 2\pi \int_0^{\pi/3} \frac{4\sin\phi}{\cos^3\phi}\, d\phi = 8\pi \cdot \frac{\sec^2\phi}{2}\Bigg|_0^{\pi/3}$$

$\left(u = \cos\phi,\ -\int u^{-3}\,du = \frac{u^{-2}}{2}\right)$

$$= 4\pi(4-1) = 12\pi$$

5) Use **Green's Theorem once** to evaluate $\int_C \vec{F}\cdot d\vec{s}$ if $\vec{F} = \langle xy, y\rangle$ and C is the closed curve that rotates once counterclockwise about the triangle with vertices (2, 0), (2, 4), and (0, 4). (6 points)



$y = -2x+4$

$$\int_C \vec{F}\cdot d\vec{s} \underset{G.T.}{=} \iint_R 0 - x \, dA$$

$$= \int_0^2\int_{4-2x}^4 -x \, dy\, dx = \int_0^2 (-x)\left[4-(4-2x)\right] dx$$

$$= \int_0^2 -2x^2\, dx = -2\,\frac{x^3}{3}\Bigg|_0^2 = \boxed{\frac{-16}{3}}$$

Note: We could also find the area of the triangle (4) using Green's Thm:
Area $= \iint 1\, dA \underset{G.T.}{=} \int_{\partial(R)} \langle 0, x\rangle \cdot d\vec{s} = \int_{\partial(R)} x\, dy = \int_0^2 x(-2dx) + \int_0^4 2\,dy + 0 = -x^2\Big|_0^2 + 8 = 4.$