Test #1 100 points Math 200 Name:

Show work

1) Evaluate each limit or show why it does not exist. Defend your answers. (~~4~~ ~~6~~ 8 points) 8 pts

1A) $\lim_{(x,y)\to(0,0)} \frac{1-e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$

Substitution gives $\frac{0}{0}$, indeterminant.

Use polar: $r \to 0$, θ is free

$\lim_{r\to 0^+} \frac{1-e^r}{r} \underset{L'H}{=} \lim_{r\to 0^+} \frac{-e^r}{1} = \boxed{-1}$

1B) $\lim_{(x,y)\to(3,3)} \frac{2x-y-3}{x-y}$

Substitution gives $\frac{0}{0}$, indeterminant.

$x=3,\ y\to 3^+ \Rightarrow \lim_{y\to 3^+} \frac{6-y-3}{3-y} = 1$

$y=3,\ x\to 3^+ \Rightarrow \lim_{x\to 3^+} \frac{2x-6}{x-3} = \lim_{x\to 3^+} 2 = 2$

$1 \neq 2 \Rightarrow$ the limit DNE

1C) $\lim_{(x,y)\to(0,0)} \frac{5-3x-4y}{x^2+y^2+3}$ ~~$= \boxed{\frac{5}{3}}$ using substitution~~

~~$\frac{5-3x-4y}{x^2+y^2+3}$ is continuous at $(0,0)$.~~

2) Find the mass of a wire with density $\delta = \frac{xy}{\sqrt{1+\cos^2(x)}}$ grams per cm if it is located on the path $y=\sin(x)$ from the point $(0,0)$ to the point $(\pi,0)$. (~~4~~ ~~8~~ points) 5 pts

$\text{Mass} = \int_C \delta\, ds$. $C: \vec{r}(t) = \langle \underset{x}{t}, \underset{y}{\sin t} \rangle,\ 0 \le t \le \pi$. $\vec{r}\,'(t) = \langle 1, \cos t \rangle$

$\Rightarrow \|\vec{r}\,'(t)\| = \sqrt{1+\cos^2 t}$.

$\therefore \text{Mass} = \int_0^\pi \frac{t \sin t}{\cancel{\sqrt{1+\cos^2 t}}} \cancel{\sqrt{1+\cos^2 t}}\, dt$

t	$\sin t$
1	$-\cos t$
0	$-\sin t$

$= \int_0^\pi t \sin t\, dt = -t\cos t + \sin t \Big|_0^\pi = \boxed{\pi}$ grams.

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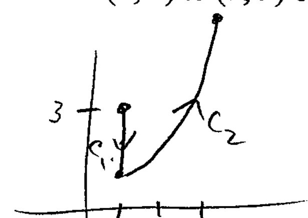
3) Find and evaluate an integral representing the work done by $\vec{F}=\langle -y, x, z+x\rangle$ on a particle that moves on the curve parameterized by $\vec{r}(t)=\langle \sin t, \sin t, \cos t\rangle$ from $t=0$ to $t=\pi$. (~~7~~ 5 points)

$$W=\int_C \vec{F}\cdot d\vec{s}\ ;\quad \vec{r}\,'(t)=\langle \cos t, \cos t, -\sin t\rangle\ ;\quad d\vec{s}=\vec{r}\,'(t)\,dt.$$

$$\therefore W=\int_0^{\pi}\langle -\sin t, \sin t, \cos t+\sin t\rangle\cdot\langle \cos t, \cos t, -\sin t\rangle\, dt$$

$$=\int_0^{\pi} \underbrace{-\sin t\cos t+\sin t\cos t}_{0} - \underbrace{\sin t\cos t}_{=\frac{\sin(2t)}{2}\,\to\,0;\ \pi} - \underbrace{\sin^2 t}_{=\frac{1-\cos(2t)}{2}\,\to\,0,\ \pi}\, dt=\int_0^{\pi}-\frac{1}{2}\,dt=\boxed{-\frac{\pi}{2}}$$

4) Evaluate $I=\int_{C_1+C_2} 4xy\,dx+2y\,dy$ if C_1 is a line segment from (1, 3) to (1, 1) and ~~a curve from~~ C_2 is the lies on $y=x^2$ from (1, 1) to (3, 9) ~~on $y=x^2$~~. (~~7~~ 5 points)



$$C_1:\ dx=0\ ;\quad \int_{C_1}\vec{F}\cdot d\vec{s}=\int_3^1 2y\,dy=y^2\Big|_3^1=1-9=\underline{\underline{-8}}.$$

$$C_2:\ \vec{\alpha}(t)=\langle t, t^2\rangle,\ 1\le t\le 3\ \Rightarrow \int_{C_2}\vec{F}\cdot d\vec{s}=\int_1^3 (4t^3+2t^2\cdot 2t)\,dt=\frac{8t^4}{4}\Big|_1^3=2(81-1)=\underline{\underline{160}}.$$

$$\therefore \int_{C_1+C_2}\vec{F}\cdot d\vec{s}=-8+160=\boxed{152}$$

~~5) Let $\vec{f}(t)=\langle P(t),Q(t)\rangle$ and $\vec{g}(t)=\langle R(t),W(t)\rangle$. Prove that $(\vec{f}\cdot\vec{g})'=\vec{f}\,'\cdot\vec{g}+\vec{f}\cdot\vec{g}\,'$. (4 points)~~ 5 Change

$\vec{f}(t)=\langle \cos t, \tan t\rangle$

$\vec{g}(\pi)=\langle 1,2\rangle$

$\vec{g}\,'(\pi)=\langle 5,8\rangle$

Find $(\vec{f}\cdot\vec{g})'(\pi)$

$$(\vec{f}\cdot\vec{g})'(\pi)=\vec{f}\,'(\pi)\cdot\vec{g}(\pi)+\vec{f}(\pi)\cdot\vec{g}\,'(\pi)$$

$$=\langle -\sin(\pi), \sec^2(\pi)\rangle\cdot\langle 1,2\rangle+\langle \cos(\pi), \tan(\pi)\rangle\cdot\langle 5,8\rangle$$

$$=\langle 0, 1\rangle\cdot\langle 1,2\rangle+\langle -1, 0\rangle\cdot\langle 5,8\rangle$$

$$=2-5=\boxed{-3}$$

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~~[illegible]~~ ok

6A) Find all ~~the~~ second partial derivatives of $f(x,y,z)=e^x y-z^2$. (~~6~~ 5 points)

$$\boxed{\begin{array}{ll} f_{xx}=e^x y & f_{xy}=f_{yx}=e^x \\ f_{yy}=0 & f_{xz}=f_{zx}=0 \\ f_{zz}=-2 & f_{yz}=f_{zy}=0 \end{array}}$$

Note: $f_x = e^x y$

$f_y = e^x$

$f_z = -2z$

6B) What is $\nabla f(0,1,2)$ if $f(x,y,z)=e^x y-z^2$? (2 points)

$$\nabla f(0,1,2) = \langle f_x, f_y, f_z\rangle \Big|_{(0,1,2)} = \boxed{\langle 1,1,-4\rangle}$$

7A

~~6C)~~ 7A) What is the divergence of ~~$f(x,y,z)=e^x y-z^2$?~~ (~~2~~ 1 points) $\vec{F}(x,y,z) = \langle e^x y, z^2, x^2\rangle$.

$$\nabla\cdot\vec{F} = \frac{\partial(e^x y)}{\partial x} + \frac{\partial(z^2)}{\partial y} + \frac{\partial(x^2)}{\partial z} = \boxed{e^x y}$$

7B

~~6D)~~ 7B) What is the curl of ~~$f(x,y,z)=e^x y-z^2$?~~ (2 points) $\vec{F} = \langle e^x y, z^2, x^2\rangle$.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x y & z^2 & x^2 \end{vmatrix} = \langle 0-2z, 0-2x, 0-e^x\rangle$$

$$\Rightarrow \boxed{\nabla\times\vec{F} = \langle -2z, -2x, -e^x\rangle}$$

~~7)~~ 8) Find the equation of the plane that contains the points P = (1,1,0), Q = (1,0,1), and R = (0,1,2). Write your final answer in standard form. (6 points)

$$\begin{array}{rl} & \hat{\imath}\ \ \hat{\jmath}\ \ \hat{k} \\ \overrightarrow{PQ} = & \langle 0,-1,1\rangle \\ \times\ \overrightarrow{PR} = & \langle -1,0,2\rangle \\ \hline & \langle -2,-1,-1\rangle \end{array}$$

Use $\vec{n} = \langle 2,1,1\rangle$

$\langle 2,1,1\rangle \cdot \langle 1,1,0\rangle = 3$

$\therefore$ the equation of the plane is

$$\boxed{2x+y+z=3}$$

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9 4

~~8~~) Let $\vec{a} = \langle 1,2,2 \rangle$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$, and $\vec{c} = \langle = -2,1,-1 \rangle$. (~~21~~ points @ ~~3~~ points each)

5 ~~8~~A) Find the area of the parallelogram spanned by $\vec{b}$ and $\vec{c}$.

$$\begin{array}{rl} \vec{b} = & 2 \;\; -1 \;\; 3 \\ \times\ \vec{c} = & -2 \;\; 1 \;\; -1 \\ \hline & \langle -2, -4, 0 \rangle \end{array}$$

$\Rightarrow$ area $= \sqrt{4+16} = \boxed{2\sqrt{5}}$ area units.

4 ~~8~~B) Find the work done on an object with displacement $\vec{a}$ by force $\vec{b}$.

$W = \vec{a} \cdot \vec{b} = \langle 1,2,2 \rangle \cdot \langle 2,-1,3 \rangle$

$= 2 - 2 + 6 = \boxed{6}$ work units

9 ~~8~~C) Find the flux of $\vec{a}$ through the parallelogram spanned by $\vec{b}$ and $\vec{c}$.

Flux $= \vec{a} \cdot (\vec{b} \times \vec{c})$

From part A

$= \langle 1,2,2 \rangle \cdot \langle -2,-4,0 \rangle$

$= -2 - 8 = -10$

9 ~~8~~D) Find the volume of the box spanned by $\vec{a}$, $\vec{b}$, and $\vec{c}$.

Vol $= |\text{Flux}| = 10$

9 ~~8~~E) Find $\vec{a}_{\vec{b}\parallel}$, the projection of $\vec{a}$ onto $\vec{b}$

$$\vec{a}_{\vec{b}\parallel} = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b} = \frac{6}{4+1+9} \langle 2,-1,3 \rangle$$

$$= \boxed{\frac{3}{7} \langle 2,-1,3 \rangle}$$

9 ~~8~~F) Find $\vec{a}_{\vec{b}\perp}$, the normal component of $\vec{a}$ to $\vec{b}$

$$\vec{a}_{\vec{b}\perp} = \vec{a} - \vec{a}_{\vec{b}\parallel}$$

$$= \langle 1,2,2 \rangle - \frac{\langle 6,-3,9 \rangle}{7}$$

$$= \boxed{\frac{\langle 1,17,5 \rangle}{7}}$$

9 ~~8~~G) Find the curvature at $r(0) = (1,0,-1)$ if $\vec{r}'(0) = \vec{b}$ and $\vec{r}''(0) = \vec{c}$.

$$\kappa = \frac{\|\vec{b} \times \vec{c}\|}{\|\vec{b}\|^3} = \frac{\|\langle -2,-4,0 \rangle\|}{\|\langle 2,-1,3 \rangle\|^3} = \frac{2\sqrt{5}}{14\sqrt{14}} = \boxed{\frac{\sqrt{5}}{7\sqrt{14}}}$$

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10

~~9~~) Find the arc length of $\vec{r}(t)=\langle \cos t, \cosh t, \sin t\rangle$ from $t = 0$ to $t = \ln(3)$. (7 points)

Hint: $\cosh^2 - \sinh^2 = 1$

$$\vec{r}\,'(t) = \langle -\sin t, \sinh t, \cos t\rangle \Rightarrow \|\vec{r}\,'(t)\| = \sqrt{\sin^2 t + \sinh^2 t + \cos^2 t}$$

$$= \sqrt{1 + \sinh^2 t} = \sqrt{\cosh^2 t}$$

$$= \underline{\cosh t}.$$

$$\therefore S = \int_0^{\ln(3)} \cosh(t)\, dt = \sinh(t)\Big|_0^{\ln(3)}$$

$$= \boxed{\sinh(\ln(3))} = \frac{3-\frac{1}{3}}{2} = \boxed{\frac{4}{3}}$$

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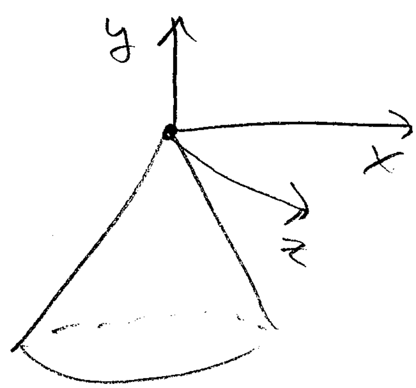
~~10~~) Sketch the surface corresponding to $y=-\sqrt{z^2+x^2}$. Label the axes. (4 points)



Lower cone about the y-axis.

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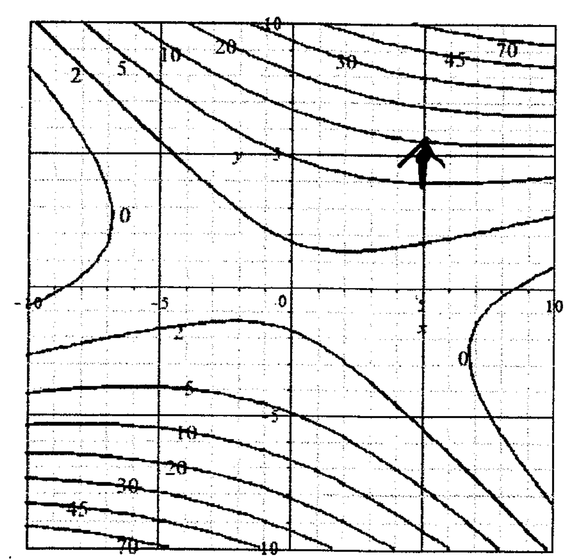
~~11~~) Use the contour plot of f(x, y) to estimate $f_y(5,5)$. Draw an appropriate segment. (3 points)



$$f_y(5,5) \approx \frac{10-5}{1.5}$$

$$= \frac{5}{\frac{3}{2}} = \boxed{\frac{10}{3}}$$

(14)

~~12~~ 13) Find a parameterization in both vector form and coordinate equations form for the line that passes through the points P = (1, 2, 1) and Q = (2, 1, 2). (~~7~~ 5 points)

$$\vec{\alpha}(t) = (1-t)\langle 1,2,1\rangle + t\langle 2,1,2\rangle$$

$$\Rightarrow \boxed{\vec{\alpha}(t) = \langle 1+t,\ 2-t,\ 1+t\rangle} \quad \text{or} \quad \boxed{\begin{aligned} x &= 1+t \\ y &= 2-t \\ z &= 1+t \end{aligned}}$$

~~13~~ 14) Convert $Q = \left\langle \overset{\rho}{2\sqrt{2}}, \overset{\theta}{\frac{\pi}{4}}, \overset{\phi}{\frac{\pi}{2}} \right\rangle_s$ to rectangular coordinates. (~~3~~ 4 points)

$\theta = \frac{\pi}{4}$

$r = \rho\sin\phi = 2\sqrt{2}\sin(\frac{\pi}{2}) = 2\sqrt{2}$

$\Rightarrow x = 2\sqrt{2}\cos\frac{\pi}{4} = 2\sqrt{2}\cdot\frac{\sqrt{2}}{2} = 2$

$y = 2\sqrt{2}\sin(\frac{\pi}{4}) = 2$

$z = \rho\cos\phi = 2\sqrt{2}\cos(\frac{\pi}{2}) = 0.$

$$\boxed{\therefore Q = (2,2,0)}$$

~~14~~ 15) ~~Write $x^2 + y^2 + (z-2)^2 = 4$ using cylindrical coordinates and simplify.~~ (~~3~~ 4 points) Sketch region ~~corresponding~~ for which ~~r≤z~~ $r \le z$ and $\rho \le 1$.

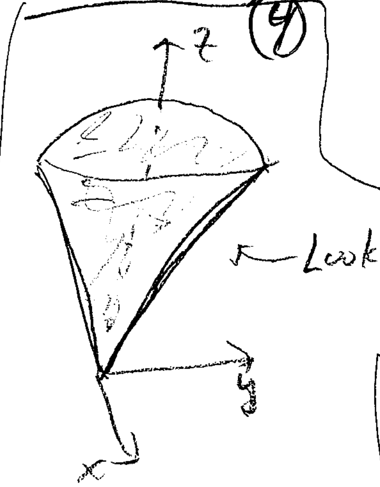
$\cancel{x^2 + (y-4)^2 = 4}$

$r \le z \Rightarrow \rho\sin\phi \le \rho\cos\phi$

$\Rightarrow \tan\phi \le 1$

$\Rightarrow \phi \le \frac{\pi}{4}$, a solid cone.

$\rho \le 1$ is the solid unit sphere



← Looks like an ice cream cone.

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