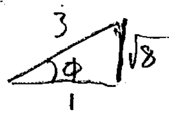
Test 1 100 points Math 200 Name: ____________________

1. (8 points) Find the equation of the plane in standard form that contains the points $(1, 2, 3)$, $(2, \pi/2, -1)_C$, and $(3, \pi/4, \arccos(1/3))_S$. Hint: first convert points to Cartesian coordinates.

$$(2, \tfrac{\pi}{2}, -1)_C = (0, 2, -1) \quad ; \quad (3, \tfrac{\pi}{4}, \cos^{-1}(\tfrac{1}{3}))_S = \left(\tfrac{3\sqrt{8}}{3}, \tfrac{\pi}{4}, 3\cdot\tfrac{1}{3}\right)_C = (2, 2, 1)$$



$$\vec{v} = \langle 1,2,3\rangle - \langle 0,2,-1\rangle = \langle 1,0,4\rangle$$

$$\times \quad \vec{w} = \langle 2,2,1\rangle - \langle 1,2,3\rangle = \langle 1,0,-2\rangle$$

$$\langle 0,6,0\rangle$$

use $\vec{n} = \langle 0,1,0\rangle$ for the normal to the plane.

Then $\vec{n}\cdot\langle x,y,z\rangle = \vec{n}\cdot\langle 1,2,3\rangle \Rightarrow \boxed{y = 2}$

2. Let $f(x, y) = \sin(x + y^2) + x^2 - 1$. You do not need to repeat work.

(a) (5 points) Find the gradient of $f(x, y)$ at $(\pi, 0)$; that is, find $\nabla f(\pi, 0)$.

$$f_x = \cos(x+y^2) + 2x$$
$$f_y = 2y\cos(x+y^2)$$
$$\Rightarrow \boxed{\nabla f(\pi, 0) = \langle -1+2\pi, 0\rangle}$$

(b) (6 points) Find and simplify the second partial derivatives of $f(x, y)$.

$$\boxed{\begin{aligned} f_{xx} &= -\sin(x+y^2) + 2 \\ f_{xy} &= -2y\sin(x+y^2) = f_{yx} \\ f_{yy} &= 2\cos(x+y^2) - 4y^2\sin(x+y^2) \end{aligned}}$$

T12

3. Let $\vec{u} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$, $\vec{v} = < \cancel{0}\, 3, 1, 2 >$, and $\vec{w} =< 1, -1, 1 >$. You may use work from one part in other parts.

(a) (4 points) Find the area of the triangle spanned by $\vec{u}$ and $\vec{w}$.

$$\text{Area} = \frac{\|\vec{u} \times \vec{w}\|}{2} = \frac{\sqrt{1+4+1}}{2} = \boxed{\frac{\sqrt{6}}{2}}$$

$$\begin{array}{rl} u = & \langle 1, -2, 3 \rangle \\ \times\, w = & \langle 1, -1, 1 \rangle \\ \hline & \langle 1, +2, 1 \rangle \end{array}$$

(b) (4 points) Find the flux of $\vec{v}$ through the parallelogram oriented from and spanned by $\vec{u}$ and $\vec{w}$. ~~and oriented from $\vec{u}$ to $\vec{w}$~~

$$\vec{v} \cdot (\vec{u} \times \vec{w}) = \langle 3, 1, 2 \rangle \cdot \langle 1, 2, 1 \rangle = \boxed{7}$$

(c) (4 points) Find $\vec{u}_{\|\vec{w}}$.

$$\vec{u}_{\|\vec{w}} = \frac{\vec{u} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} = \frac{\langle 1, -2, 3 \rangle \cdot \langle 1, -1, 1 \rangle}{3} \vec{w} = 2\vec{w} = \boxed{\langle 2, -2, 2 \rangle}$$

(d) (4 points) Find $\vec{u}_{\perp\vec{w}}$.

$$\vec{u}_{\perp\vec{w}} = \vec{u} - \vec{u}_{\|\vec{w}} = \langle 1, -2, 3 \rangle - \langle 2, -2, 2 \rangle$$

$$= \boxed{\langle -1, 0, 1 \rangle}$$

TB

4. Let $\vec{\alpha}(t) = \left\langle \cos(t), \sin(t), \frac{2(1+t)^{3/2}}{3} \right\rangle$ for all parts of this problem. You do not need to repeat work.

(a) (6 points) Find the arc length of the trace of $\vec{\alpha}(t)$ from $t = -1$ to $t = 2$.

$\vec{\alpha}'(t) = \langle -\sin(t), \cos(t), (1+t)^{1/2} \rangle = \|\vec{\alpha}'(t)\| = \sqrt{1 + 1 + t}$

$S = \int_{-1}^{2} \sqrt{2+t}\, dt = \frac{2}{3}(2+t)^{3/2} \Big|_{-1}^{2}$

$= \frac{2}{3}(2^3 - 1) = \boxed{\frac{14}{3}}$

(b) (6 points) Find the equation of the osculating plane for the trace of $\vec{\alpha}(t)$ at $t = 0$. Write the answer in $ax + by + cz = d$ form.

$\alpha(0) = (1, 0, \frac{2}{3})$ is a point in the plane.

$\vec{\alpha}'(0) = \langle 0, 1, 1 \rangle$ is parallel to the plane.

$\vec{\alpha}''(0) = \langle -\cos(0), -\sin(0), \frac{1}{2}(1+0)^{-1/2} \rangle = \langle -1, 0, \frac{1}{2} \rangle$ is parallel to plane.

$\langle 0, 1, 1 \rangle \times \langle -1, 0, \frac{1}{2} \rangle = \langle \frac{1}{2}, -1, 1 \rangle \Rightarrow \langle 1, -2, 2 \rangle$ is $\perp$ to plane. $\therefore$ the equation is

$x - 2y + 2z = \langle 1, -2, 2 \rangle \cdot \langle 1, 0, \frac{2}{3} \rangle$

$\Rightarrow \boxed{x - 2y + 2z = \frac{7}{3}}$ or $\boxed{3x - 6y + 6z = 7}$

(c) (4 points) Find the curvature for the trace of $\vec{\alpha}(t)$ at $t = 0$. Simplify your final answer.

$\kappa(0) = \frac{\|\langle \frac{1}{2}, -1, 1 \rangle\|}{\|\langle 0, 1, 1 \rangle\|^3} = \frac{\frac{1}{2} \cdot 3}{2\sqrt{2}} = \boxed{\frac{3}{4\sqrt{2}}}$ or $\boxed{\frac{3\sqrt{2}}{8}}$

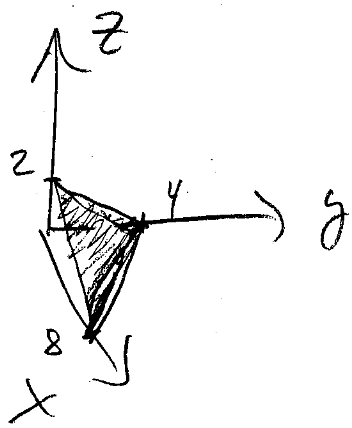
$\downarrow$

T14

5. (6 points) Write a position function with input t seconds for a particle that moves once around a circle every 4 seconds if the circle is in the plane $y = 3$ with center at $(1, 3, 5)$ and radius 6.

$$\vec{s}(t) = \langle 6\sin(\tfrac{\pi}{2}t) + 1,\ 3,\ 6\cos(\tfrac{\pi}{2}t) + 5 \rangle$$

6. (4 points) Sketch the part of the plane $x + 2y + 4z = 8$ in the first octant (x, y, and z are all non-negative) with labeled positively oriented xyz-axes.



7. (6 points) Find $\vec{f}'(0)$ if $\vec{f}(t) = \vec{p}(t) \times \langle \ln(t+1), \cos(3\pi t), e^t \rangle$, $\vec{p}(0) = \hat{\imath}$, and $\left.\frac{d\vec{p}}{dt}\right|_{t=0} = \langle -1, 0, 2 \rangle$. Show work to defend your answer.

$$\vec{f}'(0) = \vec{p}'(0) \times \langle 0, 1, 1 \rangle + \vec{p}(0) \times \langle \tfrac{1}{0+1}, -3\pi\sin(0), e^0 \rangle$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} + \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \langle -2, 1, -1 \rangle + \langle 0, -1, 0 \rangle$$

$$= \boxed{\langle -2, 0, -1 \rangle}$$

T15

8. (10 points) Find the mass of the wire that lies on a line segment from $(1, 2, 2)$ to $(3, 1, 0)$ if the density of the wire is $\delta(x, y, z) = x + y + z$. grams/cm

$\vec{s}(t) = \langle 1, 2, 2\rangle + t\langle 2, -1, -2\rangle, \quad 0 \le t \le 1 \Rightarrow \vec{s}\,'(t) = \langle 2, -1, -2\rangle$

$\Rightarrow \|\vec{s}\,'(t)\| = \sqrt{4+1+4} = 3.$ $\quad \vec{s}(t) = \langle \underset{x}{1+2t}, \underset{y}{2-t}, \underset{z}{2-2t}\rangle$

$\therefore \text{Mass} = \int_0^1 (1+2t+2-t+2-2t)\cdot 3\,dt$

$= 3\int_0^1 5-t\,dt = 3\left(5t - \frac{t^2}{2}\right)\Big|_0^1$

$= \boxed{\frac{27}{2}}$ grams

9. Find the limit if possible, or prove that the limit does not exist. Show organized work to defend your answer.

(a) (4 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^3+y^3} = L$

$\left.\begin{matrix} y\to 0 \\ x=0 \end{matrix}\right\} \Rightarrow L = \lim_{y\to 0} \frac{0}{y^3} = \underline{\underline{0}}.$

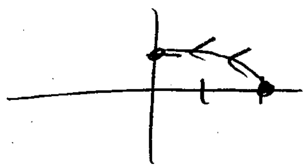
$\left.\begin{matrix} x\to 0 \\ y=x \end{matrix}\right\} \Rightarrow L = \lim_{x\to 0} \frac{x^3}{2x^3} = \underline{\underline{\frac{1}{2}}}$

$0 \ne \frac{1}{2} \Rightarrow \boxed{\nexists L \text{ DNE}}$

(b) (4 points) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}} = \lim_{r\to 0} \frac{\sin(r)}{r} = \boxed{1}$

T16

10. (10 points) Find the work done by the vector field $\vec{F}(x,y) = \overset{-2}{\cancel{-y}}\,\hat{\imath} + x\hat{\jmath}$ on a particle that moves on the ellipse $\frac{x^2}{4} + y^2 = 1$ in the first quadrant from $(2,0)$ to $(0,1)$.



$$\left.\begin{aligned} x &= 2\cos(t) \\ y &= \sin(t) \end{aligned}\right\} \Rightarrow s(t) = \langle 2\cos(t), \sin(t) \rangle$$

$$0 \le t \le \frac{\pi}{2}$$

$$\vec{s}\,'(t) = \langle -2\sin(t), \cos(t) \rangle$$

$$W = \int_0^{\pi/2} \langle -2, 2\cos(t) \rangle \cdot \langle -2\sin(t), \cos(t) \rangle \, dt$$

$$= \int_0^{\pi/2} 4\sin(t) + 2\cos^2(t)\, dt = -4\cos(t)\Big|_0^{\pi/2} + \int_0^{\pi/2} 1 + \overset{0;\ \pi}{\cancel{\cos}}(2t)\, dt$$

$$= 4 + \frac{\pi}{2} \text{ (Joules)}$$

11. (5 points) Sketch $y = z^2 + x^2$ with labeled positively oriented xyz-axes.