Test #2 100 points Math 200 Name:________________________

**Show work.**

1A) Find the tangent plane, in ax + by + cz = d form, for the graph of $f(x,y)=4\sqrt{x+y^3}$ at (3,1,$\cancel{2}$) 8. (5 points)

my error

$$f_x(3,1) = 4\cdot\tfrac{1}{2}(x+y^3)^{-1/2}\Big|_{(3,1)} = 1$$

$$f_y(3,1) = 4\cdot\tfrac{1}{2}(x+y^3)^{-1/2}\cdot 3y^2\Big|_{(3,1)} = 3$$

$$\Rightarrow z-\cancel{2}\,8 = 1(x-3)+3(y-1)$$

$$\Rightarrow \boxed{x+3y-z=4} \quad \text{or} \quad x+3y-z=-2$$

1B) Use the ~~differential~~ linear approximation for to estimate $f(2.8,1.1)$ ~~if~~ $f(x,y)=4\sqrt{x+y^3}$ ~~as in #1A~~. (3 points)

$$f(2.8,1.1) \approx 8 + 1(2.8-3)+3(1.1-1)$$

$$\Rightarrow f(2.8,1.1) \approx 8 - .2 + .3 = \boxed{8.1}$$

(understandable if 2 is used instead of 8 to get 2.1)

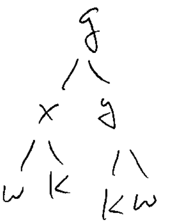
1C) Find the directional derivative of $f(x,y)=4\sqrt{x+y^3}$ at (3,1) in the direction $\langle 3,4\rangle$. (3 points)

$$\nabla f(3,1)\cdot\frac{\langle 3,4\rangle}{5} = \frac{\langle 1,3\rangle\cdot\langle 3,4\rangle}{5} = \frac{15}{5} = \boxed{3}$$

1D) What is the maximum value of $D_{\hat{u}}f(3,1)$ if $f(x,y)=4\sqrt{x+y^3}$? Defend your answer. (4 points)

$$\|\nabla f(3,1)\| = \|\langle 1,3\rangle\| = \boxed{\sqrt{10}}$$

2) Find $\dfrac{\partial g}{\partial k}$ at $k=-2$ and $w=\pi$ if $g(x,y)=\sin(xy)$, $x=wk^2$, $y=y(k,w)$, $y(-2,\pi)=2$, $y_k(-2,\pi)=4$, and $y_w(-2,\pi)=6$. (5 points)




$$\Rightarrow \frac{\partial g}{\partial k} = \frac{\partial g}{\partial x}\Big|_{\substack{x=4\pi\\ y=2}} \cdot \frac{\partial x}{\partial k}\Big|_{\substack{k=-2\\ w=\pi}} + \frac{\partial g}{\partial y}\Big|_{\substack{x=4\pi\\ y=2}} \cdot \frac{\partial y}{\partial k}\Big|_{\substack{k=-2\\ w=\pi}}.$$

$$\left.\begin{aligned} g_x &= y\cos(xy)\\ g_y &= x\cos(xy)\\ x_k &= 2wk \end{aligned}\right\} \Rightarrow \frac{\partial g}{\partial k} = 2\cdot 1\cdot(-4\pi) + 4\pi\cdot 1\cdot 4 = 8\pi$$

A

3) Suppose $f(\cancel{m,y}) = \frac{L}{R^4}$ LR where L is measured with a ~~10%~~ maximum percent error and R is measured with a 2% maximum percentage error. Use differentials to estimate the maximum percentage error of $f(\cancel{m,y})$ LR when L = 5 and R = 0.1. (6 points)

$$f(5, 0.1) = \frac{5}{(.1)^4} = 5 \cdot 10^4.$$

$$\frac{\max df}{f} \cdot 100 = \left( \frac{\frac{1}{R^4} dL}{\left(\frac{L}{R^4}\right)} + \frac{\left|\frac{-4L}{R^5}\right| dR}{\left(\frac{L}{R^4}\right)} \right) \cdot 100$$

$$= \frac{dL}{L} \cdot 100 + 4 \frac{dR}{R} \cdot 100 = 3\% + 4(2\%) = \boxed{11\% \text{ max \% error}}$$

4) Find the tangent plane, in ax + by + cz = d form, for the graph of $3 + xyz = \ln(xyz) + e^3$ at the point (e, e, e). (6 points)

Let $F = xyz - \ln(xyz)$.

$\nabla F(e,e,e) = \langle e^2 - e^{-1}, e^2 - e^{-1}, e^2 - e^{-1} \rangle$,

$= (e^2 - e^{-1})\langle 1, 1, 1 \rangle$, is

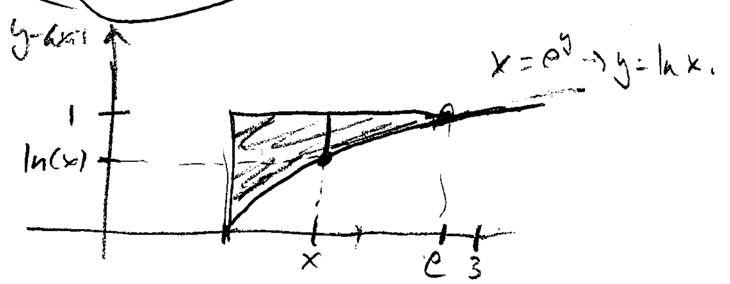
$\perp$ to T.P., so use

$\vec{n} = \langle 1, 1, 1 \rangle$.

Since $\langle 1,1,1 \rangle \cdot \langle e,e,e \rangle = 3e$, the T.P. is

$$\boxed{x + y + z = 3e}$$

5) Sketch the region of integration for $I = \int_0^1 \int_1^{e^y} f(x,y)\,dx\,dy$ and then rewrite I with the order of integration switched. to $dy\,dx$. (8 points)




$$\Rightarrow \boxed{I = \int_1^e \int_{\ln(x)}^1 f(x,y)\,dy\,dx}$$

6) Evaluate ~~the following~~ line integrals. **Use the FTCLI ~~whenever~~ possible**. If impossible, **show why** this is so before ~~using test #1 methods to find~~ finding the line integral. (14 points)

6A) $\int_C \vec{F}\cdot d\vec{s}$ if $\vec{F}=\langle x^2y, y\rangle$ and C is the curve $\mathbf{r}(t)=\langle \cancel{\sin(t)}\,t^2, \cancel{\sin(2t)}\,t\rangle$ for $0\le t\le 1$.

$\frac{\partial y}{\partial x}=0\neq x^2=\frac{\partial x^2y}{\partial y} \Rightarrow \vec{F}$ is not a gradient field.

$\therefore I=\int_0^1 \langle t^5, t\rangle\cdot\langle 2t, 1\rangle\,dt$

$=\int_0^1 2t^6+t\,dt$

$=\frac{2t^7}{7}+\frac{t^2}{2}\Big|_0^1$

$=\frac{4+7}{14}=\boxed{\frac{11}{14}}$

6B) $I=\int_C \underbrace{(3x^2y)}_{P}dx+\underbrace{(y+x^3)}_{Q}dy$ and C is the curve oriented from $(-1,-1)$ to $(2,8)$ on the path $y=x^3$.

$2y+x^3$

$\frac{\partial Q}{\partial x}=+3x^2=\frac{\partial P}{\partial y}$; Since the domain of $\vec{F}=\langle P,Q\rangle$ is $\mathbb{R}^2$ and $\mathbb{R}^2$ is S.C.,

$\vec{F}$ is a gradient field.

$\phi(a,b)=\int_0^a 0\,dx+\int_0^b 2y+a^3\,dy$

$\Rightarrow \phi(a,b)=y^2\Big|_0^b+a^3b=b^2+a^3b$

$\therefore I=\phi(2,8)-\phi(-1,-1)$

$=64+64-(1+1)$

$=\boxed{126}$

7) Evaluate $I=\int_0^2\int_{-z}^{z}\int_{-y}^{y} 3y+\cancel{xy}^{\,0}+z^2\,dx\,dy\,dz$. (6 points)

ODD, Even

$I=\int_0^2\int_{-z}^{z} 2\cdot(3y+z^2)x\Big|_0^y\,dy\,dz$

$=\int_0^2\int_{-z}^{z} \underbrace{6y^2}_{\text{Even}}+\underbrace{\cancel{2yz^2}^{\,0}}_{\text{ODD}}\,dy\,dz$

$=\int_0^2 2\cdot\frac{6y^3}{3}\Big|_0^z\,dz$

$=\int_0^2 4z^3\,dz$

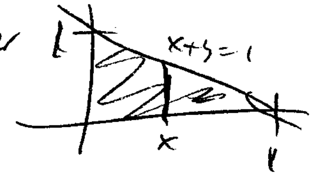
$=\frac{4z^4}{4}\Big|_0^2=\boxed{16}$

$z = 15\sqrt{y}$



**8) Use a double integral** to find the volume of the solid bounded by z = 0, x = 0, y = 0, x + y = 1, and $z = 4x^2 + y$. (8 points)

$z = 2y$ is the top, and always nonnegative over




$$\therefore \text{Vol} = \int_0^1 \int_0^{1-x} 15\sqrt{y}\, dy\, dx$$

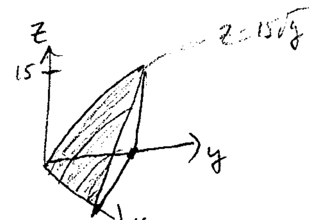
$$= \int_0^1 \left(10 y^{\frac{3}{2}} \Big|_0^{1-x}\right) dx$$

$$= \int_0^1 10(1-x)^{\frac{3}{2}}\, dx$$

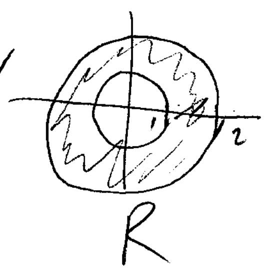
$$\begin{pmatrix} u = 1-x \\ du = -dx \end{pmatrix}$$

$$\text{Vol} = -4(1-x)^{\frac{5}{2}} \Big|_0^1 = 0 + 4$$

$$\Rightarrow \boxed{\text{Vol} = 4} \text{ cubic units.}$$




9) Find the mass of the lamina that lies in the region $1 \le x^2 + y^2 \le 4$ with density $\delta(x, y) = x^2$. (8 points)




$$\text{MASS} = \iint_R \delta\, dA$$

$$\Rightarrow \text{Mass} = \int_0^{2\pi} \int_1^2 r^2 \cos^2\theta \cdot r\, dr\, d\theta$$

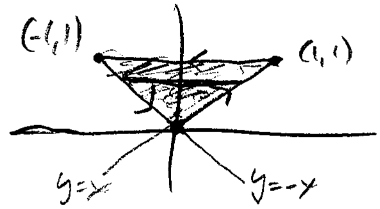
$$= \int_0^{2\pi} \cos^2\theta\, d\theta \cdot \int_1^2 r^3\, dr$$

$$= \int_0^{2\pi} \frac{1 + \cancel{\cos 2\theta}}{2}\, d\theta \cdot \left(\frac{r^4}{4}\Big|_1^2\right) \quad \to 0;\ 2\pi$$

$$= \pi \cdot \frac{16-1}{4} = \boxed{\frac{15\pi}{4}}$$

10) Use **Green's Theorem once** to evaluate $\int_C \vec{F}\cdot d\vec{s}$ if $\vec{F} = \left\langle \tan^2 x + y^2, e^{y^2} - x \right\rangle$ and C is the triangle oriented positively with vertices at (0, 0), (1, 1), and $(-1,1)$. (6 points)

$$\int_C \vec{F}\cdot d\vec{s} \underset{G.T.}{=} \iint_R (-1) - 2y \, dA$$




$$= \int_0^1 \int_{-y}^{y} \underset{(\text{even})}{-1-2y} \, dx\, dy$$

$$= \int_0^1 2(-1-2y)\, y \, dy$$

$$= \int_0^1 -2y - 4y^2 \, dy$$

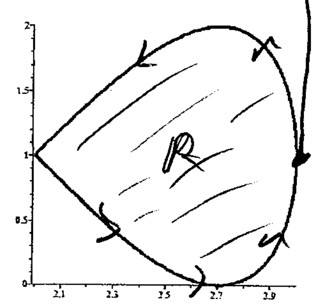
$$\therefore \int_C \vec{F}\cdot d\vec{s} = -y^2 - \frac{4y^3}{3}\Big|_0^1$$

$$= -1 - \frac{4}{3}$$

$$= \boxed{-\frac{7}{3}}$$

11) Use **Green's** Theorem to find the area of the region inside the closed curve C parameterized by $\mathbf{r}(t) = \langle 2+\sin(t), 1-\sin(2t)\rangle$ for $0 \le t \le \pi$. The graph of C is shown below. (6 points)

(once around)




$\vec{r}\,'(\frac{\pi}{2})$

$$\text{Area} = \iint_R 1 \, dA \underset{G.T.}{=} \oint_C \langle 0, x\rangle \cdot \langle \cos t, -2\cos(2t)\rangle \, dt$$

$$= \int_0^{\pi} \langle 0, 2+\sin t\rangle \cdot \langle \cos t, -2\cos(2t)\rangle \, dt$$

$$= \int_0^{\pi} \cancel{-4\cos(2t)}^{\,0;\ \pi} - 2\sin(t)\cos(2t) \, dt$$

use either:
① $\sin(a+b) = \sin a \cos b + \sin b \cos a$
or
② $\cos^2 t = \frac{1+\cos 2t}{2}$
Sol. $\sin t \cos 2t = \sin t\,(2\cos^2 t - 1)$ then $u = \cos t$, $du = -\sin t\, dt$

$$= -2\int_0^{\pi} \frac{\overset{t+2t}{\sin(3t)} + \overset{t-2t}{\sin(-t)}}{2} \, dt$$

$$= \left(\frac{\cos(3t)}{3} - \cos(t)\right)\Big|_0^{\pi} = \left(\frac{-1}{3}+1\right) - \left(\frac{1}{3}-1\right) = 2 - \frac{2}{3}$$

$$= \boxed{\frac{4}{3}}$$

$r(\frac{\pi}{2}) = (3, 1)$

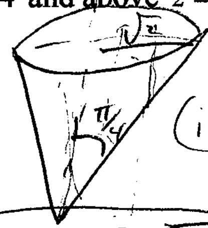
$\vec{r}\,'(\frac{\pi}{2}) = \langle \cos t, -2\cos(2t)\rangle \Big|_{t=\frac{\pi}{2}}$

$= \langle 0, 2\rangle$

$\Rightarrow Bd(R)$ is positively oriented.

12) **Use cylindrical coordinates to set up** an integral equal to $M_{xy}$, the moment about the ~~z~~– axis, for the solid that is inside $x^2+y^2+z^2=4$ and above $z=\sqrt{x^2+y^2}$. The density is $\delta(x,y,z)=z+y^2$ **Do not evaluate the integral.** (6 points)

Hint: $\bar{z}=\frac{M_{xy}}{Mass}$.

E: 

(ice cream cone)

$2x^2+2y^2=4 \Rightarrow x^2+y^2=2$

For z: min z on cone: $z=r$.

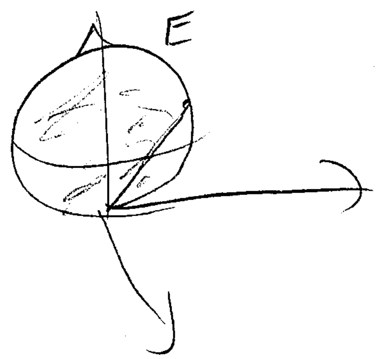
Max z on sphere: $z=\sqrt{4-r^2}$

$$\boxed{M_{xy}=\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} z\left(z+r^2\sin^2\theta\right) r\,dz\,dr\,d\theta}$$

Note: $M_{xy}=\iiint_E z\,\delta\,dV$

13) **Use spherical coordinates** to find $I_0$, the moment of inertia about (0, 0, 0), for the solid ball $x^2+y^2+(z-1)^2\le 1$ if its density is $\cancel{\delta}(x,y)=\frac{z}{(x^2+y^2+z^2)^2}$ grams per $cm^2$. (6 points)

$\delta(x,y,z)$




$x^2+y^2+(z-1)^2\le 1$

$\Rightarrow \rho^2-2z\le 0 \Rightarrow \rho\le 2\cos\phi$ Max $\rho$ Coordinate

$$I_0=\iiint_E \rho^2\,\delta(x,y,z)\,dV$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\cos\phi}\cancel{\rho^2}\cdot\frac{\rho\cos\phi}{\cancel{\rho^4}}\cancel{\rho^2}\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\frac{\rho^2}{2}\cdot\cos\phi\cdot\sin\phi\Big|_0^{2\cos\phi}d\phi\,d\theta=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}2\cos^3\phi\,\sin\phi\,d\phi\,d\theta$$

$$=2\pi\cdot\left(-\frac{\cos^4\phi}{2}\Big|_0^{\frac{\pi}{2}}\right)=\pi$$