Test 2 100 points Math 200 Name: ______________________

1. For all parts of this problem let $\vec{F}(x,y,z) = (6x^2y)\hat{\imath} + (mx^3 + 2y)\hat{\jmath} + (\sin(z))\hat{k}$.

(a) ($\cancel{6}$ 4 points) Find m so that $\vec{F}(x,y,z)$ is a gradient field.

$$\frac{\partial(mx^3+2y)}{\partial x} = \frac{\partial(6x^2y)}{\partial y} \Rightarrow 3mx^2 = 6x^2 \Rightarrow \boxed{m=2}$$

(b) ($\cancel{8}$ 10 points) Use the FTCLI to evaluate $I = \int_C \vec{F}(x,y,z)\cdot d\vec{s}$ if C is parameterized by $\vec{\alpha}(t) = \langle t^2, t\sin(t), t\cos(t)\rangle$ for $0 \le t \le \cancel{\pi/2}$ π

$$\phi = \int 6x^2y\,dx = 2x^3y + c(y,z) \,;\quad \phi_y = 2x^3 + c_y = 2x^3 + 2y \Rightarrow c_y = 2y.$$

$$\therefore c(y,z) = y^2 + c_1(z). \quad \phi_z = c_1'(z) = \sin(z) \Rightarrow c_1(z) = -\cos(z) + \cancel{c_2}^{0}.$$

$$\therefore \phi(x,y,z) = 2x^3y + y^2 - \cos(z)$$

$$\therefore I = \phi(\alpha(\pi)) - \phi(\alpha(0)) = \phi(\pi^2, 0, -\pi) - \phi(0,0,0)$$

$$\Rightarrow I = (0+0+1) - (0+0-1) \Rightarrow \boxed{I=2}$$

Room

2. (6 points) Evaluate $I = \int_0^{\pi/2}\int_0^y\int_0^x \cos(\cancel{x+}\,y+z)\,dz\,dx\,dy$.

$$I = \int_0^{\pi/2}\int_0^y \sin(y+z)\Big|_0^x\,dx\,dy = \int_0^{\pi/2}\int_0^y \sin(x+y) - \sin(y)\,dx\,dy$$

	y	$\sin y$
+		
−	1	$-\cos y$
	0	$-\sin(y)$

Room

$$= \int_0^{\pi/2} -\cos(x+y) - x\sin(y)\Big|_0^y\,dy$$

$$= \int_0^{\pi/2} -\cancel{\cos(2y)}^{0;\,\pi} - y\sin(y) + \cos(y)\,dy$$

$$= y\cos(y) - \sin(y) + \sin(y)\Big|_0^{\pi/2} = \boxed{0}$$

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3. For all parts of this problem, $f(x,y)$ is a differentiable function with $f(1,3)=4$, $\nabla f(1,3) = -2\hat{\imath}+5\hat{\jmath}$, and $P=(1,3,4)$.

(a) (4 points) Find the equation of the tangent plane in standard form to the graph of $z=f(x,y)$ at the point P.

$$z-4 = (-2)(x-1) + 5(y-3)$$

$$\Rightarrow \quad 2x-5y+z = 4+2-15$$

$$\Rightarrow \quad \boxed{2x-5y+z=-9}$$

check: $2\cdot 1 - 5\cdot 3 + 4 = -9$ ✓

(b) (4 points) Use the linear approximation for $f(x,y)$ at the point P to estimate $f(1.2, 2.7)$.

$$f(1.2, 2.7) \approx 4 + (-2)(.2) + 5(-.3)$$

$$= 4 - .4 - 1.5$$

$$= \boxed{2.1}$$

(c) (3 points) Find the minimum value of $D_{\vec{u}}f(3,1)$. Show some work.

$$\text{Min value} = -\|\nabla f\| = -\sqrt{4+25} = \boxed{-\sqrt{29}}$$

↑

4. (6 points) Let $F(x,y)$ and $G(t)$ be differentiable functions so that

$$F(4,3)=1 \quad F_x(4,3)=2 \quad F_y(4,3)=3 \quad \cancel{G(2)=-3} \quad \cancel{G'(2)=4}$$

Let $h(x,y) = F(x^2y, x+y) \cancel{+G(xy)}$. Find $\frac{\partial h}{\partial y}(2,1)$. Show organized work.

h
$\frac{\partial h}{\partial F}=1$
F
$x^2y \quad x+y$
$x \quad y \quad x \quad y$

$$\Rightarrow \quad \left.\frac{\partial h}{\partial y}\right|_{(2,1)} = \left.\frac{\partial h}{\partial F}\cdot\frac{\partial F}{\partial x^2y}\cdot\frac{\partial x^2y}{\partial y} + \frac{\partial h}{\partial F}\cdot\frac{\partial F}{\partial(x+y)}\cdot\frac{\partial(x+y)}{\partial y}\right|_{(2,1)=(x,y)}$$

$$= \left. 1\cdot F_x(x^2y, x+y)\cdot x^2 + 1\cdot F_y(x^2y, x+y)\cdot 1\right|_{(2,1)}$$

$$= 1\cdot F_x(4,3)\cdot 4 + 1\cdot F_y(4,3)\cdot 1$$

$$= 1\cdot 2\cdot 4 + 1\cdot 3\cdot 1 = 11$$

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$\cancel{*}$5. (6 ~~points~~) $I(M,H) = \frac{\cancel{L}\ M}{H^2}$ is the body mass index for a person with mass M and height H. If $M = \cancel{84}\ 80 \pm 2$ kg and $H = 2 \pm 0.1$ m, use differentials to estimate the maximum percentage error.

$$\max \frac{dI}{I} = \frac{\frac{1}{H^2}dM + \left|\frac{2M}{H^3}dH\right|}{I} = \frac{\frac{2}{4} + \frac{2\cdot 80}{8}(0.1)}{\left(\frac{80}{4}\right)} = \frac{\frac{1}{2}+2}{20}$$

$$\therefore \frac{dI}{I}\cdot 100 = \frac{\left(\frac{1}{2}+2\right)}{\cancel{20}}\cancel{100}^{5} = \boxed{12.5\%}$$

6. (6 points) Find the tangent plane in standard form for the graph of $2x + 3y - 5z = 2\sin(x) + \sin(y) + \sin(z)$ at the point (π, π, π).

Let $f(x,y,z) = 2x + 3y - 5z - 2\sin(x) - \sin(y) - \sin(z)$

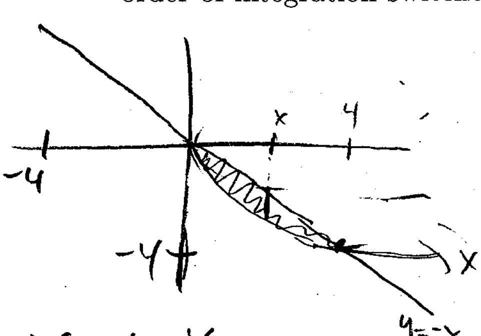
$\Rightarrow \nabla f(\pi,\pi,\pi) = \langle 2 - 2\cos(\pi),\ 3 - \cos(\pi),\ -5 - \cos(\pi)\rangle$

$= \langle 4, 4, -4\rangle$. Use $\vec{n} = \langle 1, 1, -1\rangle$, so

$x + y - z = \langle 1,1,-1\rangle \cdot \langle \pi,\pi,\pi\rangle \Rightarrow \boxed{x + y - z = \pi}$

?
6

7. ($\cancel{8}$ points) Sketch the region of integration for $I = \int_{-4}^{0}\int_{y^2/\cancel{8}\ 4}^{-y} f(x,y)\,dx\,dy$ and then rewrite I with the order of integration switched to $dy\,dx$.



$\boxed{I = \int_0^4 \int_{-2\sqrt{x}}^{-x} f(x,y)\,dx\,dy}$

$x = \frac{y^2}{4} \Rightarrow y = -2\sqrt{x}$

$0 \le y \le -4$

$\frac{y^2}{4} \le x \le -y$

(18)

8

net

$8x\sin(xy)\cos(xy)$

8. ~~(6~~ points) Find the volume of the solid that is below the graph of $z = \cancel{x\sin(x+y)}$ ~~and above $z=0$~~ if $0 \le x \le \cancel{\pi/6}\ 1$ and $0 \le y \le \cancel{\pi/3}\ \pi$

$$\text{Vol} = \int_0^1 \int_0^\pi 2x\sin(xy)\cos(xy)\,dy\,dx \leftarrow 2 \begin{cases} u = \sin(xy) \\ du = x\cos(xy)\,dy \end{cases} \Rightarrow \int 2u\,du = \underline{u^2+c}$$

$$= \int_0^1 \sin^2(xy)\Big|_0^\pi dx$$

$$= \int_0^1 \sin^2(\pi x) - 0\,dx$$

$$= \int_0^1 \frac{1-\cancel{\cos(2\pi x)}}{2}\,dx = \boxed{\frac{1}{2}}$$

$\rightarrow 0;\ 1T$

Or $\text{Vol} = \int_0^1 \int_0^\pi x\sin(2xy)\,dy\,dx$

$u = 2xy$
$du = 2x$

$$= \int_0^1 \int_0^{2\pi x} \frac{1}{2}\sin(u)\,du\,dx$$

$$= \int_0^1 -\frac{1}{2}\cos(u)\Big|_0^{2\pi x} dx$$

$$= -\frac{1}{2}\int_0^1 \cancel{\cos(2\pi x)} - 1\,dx$$

$\rightarrow 0;\ \pi$

$$= \boxed{\frac{1}{2}}$$

solid $\quad x^2+y^2 \le 4$

9. (8 points) Find the mass of the part of the cylinder $x^2+y^2=4$ for which $x \ge 0$, $y \ge 0$, and $0 \le z \le 3$ if the density is $\delta(x,y) = z + x^2y$. kg/m³

$$\text{Mass} = \int_0^{\pi/2}\int_0^2\int_0^3 (z+x^2y)\cdot r\,dz\,dr\,d\theta \;;\quad z+x^2y = z + r^3\cos^2\theta\sin\theta$$

$$= \int_0^{\pi/2}\int_0^2\int_0^3 zr + r^4\cos^2\theta\sin\theta\,dz\,dr\,d\theta$$

$$= \int_0^{\pi/2} d\theta \cdot \int_0^2 r\,dr \cdot \int_0^3 z\,dz + \int_0^{\pi/2}\cos^2\theta\sin\theta\,d\theta \cdot \int_0^2 r^4\,dr \cdot \int_0^3 dz$$

$$= \frac{\pi}{2}\cdot\left(\frac{r^2}{2}\Big|_0^2\cdot\frac{z^2}{2}\Big|_0^3\right) + \left(-\frac{\cos^3\theta}{3}\Big|_0^{\pi/2}\right)\cdot\left(\frac{r^5}{5}\Big|_0^2\right)\cdot 3$$

$$= \frac{\pi}{2}\cdot 2\cdot\frac{9}{2} + \frac{1}{\cancel{3}}\cdot\frac{32}{5}\cdot\cancel{3}$$

$$= \boxed{\frac{9\pi}{2} + \frac{32}{5}}\ \text{kg}$$

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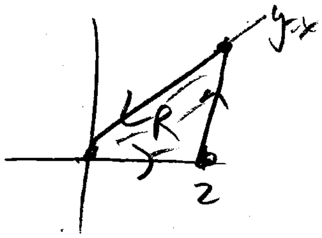
~~14~~

+2 ?

10. (~~6~~ 8 points) Use Green's Theorem exactly once to evaluate $I = \int_C x^2y^3\,dx + \sin(y)\,dy$ if C is the closed curve that rotates once counter-clockwise about the triangle with vertices at (0, 0), (2,0), and (2,2) in the xy-plane.

$$I = \iint_R 0 - 3x^2y^2\,dA = \int_0^2\int_0^x -3x^2y^2\,dy\,dx$$



$$= \int_0^2 \left(-x^2y^3\right)\Big|_0^x\,dx$$

$$= \int_0^2 -x^5\,dx$$

$$= -\frac{x^6}{6}\Big|_0^2 = \boxed{-\frac{32}{3}}$$

11. (~~6~~ 7 points) Use Green's Theorem exactly once to ~~find the area of the region inside the closed curve C parameterized once around by $\vec{\alpha} = \langle 2+\sin(t), 1-\sin(2t)\rangle$ for $0 \le t \le \pi$.~~ the formula for the area ~~of an ell~~ inside an ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1$. ~~You need a picture~~

$$\text{Area} = \iint_{R:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} 1\,dA \overset{G.T.}{=} \oint_{Bd(R)} \langle 0, x\rangle \cdot d\vec{s},$$

$Bd(R)$: $\vec{\alpha}(t) = \langle a\cos t, b\sin t\rangle \Rightarrow \vec{\alpha}'(t) = \langle -a\sin t, b\cos t\rangle$, $0 \le t \le 2\pi$

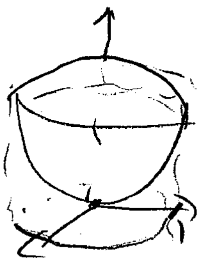
$$\therefore \text{Area} = \int_0^{2\pi} \langle 0, a\cos(t)\rangle \cdot \vec{\alpha}'(t)\,dt$$

$$= \int_0^{2\pi} ab\cos^2(t)\,dt = ab\int_0^{2\pi} \frac{1+\cos(2t)}{2}\,dt$$

$$= \boxed{\pi ab}.$$

~~12~~ 15

12. (8 points) Use cylindrical coordinates to find the volume of the solid E that is the intersection of the solid parabola $z \geq x^2 + y^2$ and the solid ball $x^2 + y^2 + z^2 \leq 2$.



Intersection:

$z = x^2+y^2$

$x^2+y^2+z^2 = 2$

$\Rightarrow z + z^2 = 2$

$\Rightarrow z^2+z-2=0$

$(z+2)(z-1)=0$

$z=-2,\ z=1.$

$z>0 \Rightarrow \underline{z=1}$, so $1 = x^2+y^2$

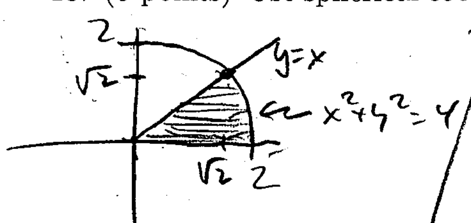
$$Vol = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} 1 \cdot r\,dz\,dr\,d\theta$$

$$= 2\pi \cdot \int_0^1 r\sqrt{2-r^2} - r^3\,dr$$

$$= 2\pi\left[\frac{-1}{3}(2-r^2)^{3/2} - \frac{r^4}{4}\right|_0^1$$

$$= 2\pi\left[-\frac{1}{3}\left(1-2^{3/2}\right) - \frac{1}{4}\right] = \boxed{\pi\left[\frac{4\sqrt{2}}{3} - \frac{7}{6}\right]}$$

13. (6 points) Use spherical coordinates to evaluate $I = \displaystyle\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}}\int_0^{\sqrt{4-x^2-y^2}} \frac{1}{x^2+y^2+z^2}\,dz\,dx\,dy.$

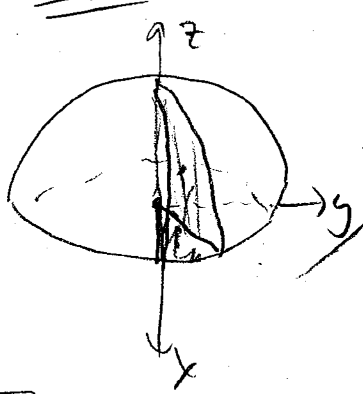


Intersection:

$x^2+x^2=4$

$\Rightarrow x^2 = 2$

$\Rightarrow \underline{(\sqrt{2},\sqrt{2})},$



$$\Rightarrow I = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^2 \frac{1}{\rho^2}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\Rightarrow I = \pi/4 \cdot \int_0^{\pi/2}\sin\phi\,d\phi \cdot 2$$

$$\Rightarrow I = \pi/2 \cdot \left(-\cos\phi\right|_0^{\pi/2}$$

$$\Rightarrow I = \pi/2\,(0+1) = \boxed{\frac{\pi}{2}}$$

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