Test 2 100 points Math 200 Name: ______________________

1. For all parts of this problem let $\vec{F}(x,y,z) = (6x^2)\hat{\imath} + (z^3 - \sin(y))\hat{\jmath} + (myz^2 - \cos(z))\hat{k}$.

(a) (4 points) Find $m$ so that $\vec{F}(x,y,z)$ is a gradient field.

$$\frac{\partial(myz^2 - \cos(z))}{\partial y} = \frac{\partial(z^3 - \sin(y))}{\partial z}$$

$$\Rightarrow mz^2 = 3z^2$$

$$\Rightarrow \boxed{m=3}$$

(b) (10 points) Use the FTCLI to evaluate $I = \int_C \vec{F}(x,y,z)\cdot d\vec{s}$ if $C$ is parameterized by $\vec{\alpha}(t) = \langle t\sin(t), t-2\pi, t\cos(t)\rangle$ for $0 \le t \le 2\pi$.

$$\phi = \int 6x^2\,dx = 2x^3 + f(y,z)$$

$$\phi = \int z^3 - \sin(y)\,\partial y = z^3y + \cos(y) + g(x,z)$$

$$\phi = \int 3yz^2 - \cos(z)\,\partial z = z^3y - \sin(z) + h(x,y)$$

guess $\Rightarrow \boxed{\phi(x,y,z) = 2x^3 + z^3y + \cos(y) - \sin(z)}$

Check: $\nabla\phi = \langle 6x^2, z^3 - \sin(y), 3z^2y - \cos(z)\rangle$ ✓

OR $\phi = 2x^3 + f(y,z) \Rightarrow \phi_y = f_y = z^3 - \sin(y)$

$\Rightarrow f(y,z) = yz^3 - \sin(y) + h(z)$

$\Rightarrow \phi = 2x^3 + yz^3 - \sin(y) + h(z)$

$\Rightarrow \phi_z = 3yz^2 + h'(z) = 3yz^2 - \cos(z) \Rightarrow h(z) = -\sin(z)$

$\therefore \phi = 2x^3 + z^3y + \cos(y) - \sin(z)$

OR $\phi(a,b,c) = \int_0^a 6a^2\,dx + \int_0^b -\sin(y)\,dy + \int_0^c 3bz^2 - \cos(z)\,dz$

$= 2a^3 + \cos(b) - 1 + bc^3 - \sin(c)$

$\phi(0,0,0) = -1 \Rightarrow \phi(x,y,z) = 2x^3 + \cos(y) + yz^3 - \sin(z)$

$$\therefore I = \phi(\alpha(2\pi)) - \phi(\alpha(0))$$

$$= \phi(0,0,2\pi) - \phi(0,-2\pi,0)$$

$$= 1 - (1) = \boxed{0}$$

2. (6 points) Evaluate $I = \int_0^1 \int_0^x \int_0^{xy} \frac{4ze^{xy}}{y^2}\,dz\,dy\,dx$.

$$= \int_0^1 \int_0^x \frac{4e^{xy}}{y^2}\left(\frac{z^2}{2}\right)\Big|_0^{xy} dy\,dx$$

$$= \int_0^1 \int_0^x 2x^2 e^{xy}\,dy\,dx$$

$$= \int_0^1 2x^2\left(\frac{e^{xy}}{x}\right)\Big|_0^x dx$$

$$= \int_0^1 2x(e^{x^2} - 1)\,dx$$

$$= e^{x^2} - x^2\Big|_0^1$$

$$= e - 1 - 1 + 0$$

$$= \boxed{e-2}$$

TZ 2

3. For all parts of this problem, $f(x, y)$ is a differentiable function with $f(3, -1) = \cancel{2}$ 6, $\nabla f(3, -1) = 4\hat{\imath} - 2\hat{\jmath}$, and $P = (3, -1, 6)$.

   (a) (4 points) Find the equation of the tangent plane in standard form to the graph of $z = f(x, y)$ at the point $P$.

$$z - 6 = 4(x-3) - 2(y+1)$$

$$\Rightarrow -6 = 4x - 2y - z - 14$$

$$\Rightarrow \boxed{4x - 2y - z = 8}$$

   (b) (4 points) Use the linear approximation for $f(x, y)$ at the point $P$ to estimate $f(2.9, -0.9)$.

$$f(2.9, -0.9) \approx f(3,-1) + 4(2.9-3) - 2(-0.9+1)$$

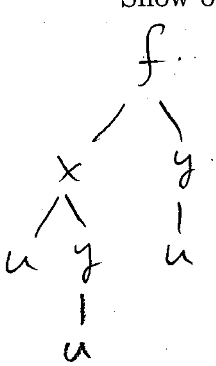
$$= 6 - .4 - .2$$

$$= \boxed{5.4}$$

   (c) (3 points) Find the minimum value of $D_{\vec{u}} f(3, -1)$. Show some work.

min value $= -\|\nabla f(3,-1)\| = -\sqrt{16+4} = \boxed{-2\sqrt{5}}$

4. (6 points) Let $f(x, y) = x^2 - 3y$, $x(u, y) = 4u + y^2$, $y = y(u)$, $y(3) = 2$, and $y'(3) = -1$. Find $\left.\dfrac{df}{du}\right|_{u=3}$. Show organized work.



$$\Rightarrow \frac{df}{du} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial y}\cdot\frac{dy}{du} + \frac{\partial f}{\partial y}\frac{dy}{du};$$

$u = 3 \Rightarrow y = 2$

$x = 12 + 4$

$= 16$

$$= 2x \cdot 4 + 2x \cdot 2y \cdot (-1) + (-3)(-1)$$

$$= 32 \cdot 4 + 32 \cdot 4(-1) + 3$$

$$= 3$$

T23

5. (6 points) Suppose $f(L,R) = \dfrac{L^2}{R^4}$ where $L$ is measured with a 3% maximum percent error and $R$ is measured with a 2% maximum percentage error. Use differentials to estimate the maximum percentage error of $f(L,R)$ when $L = 5$ and $R = 1$.

$$\text{max \% error} \approx \frac{\left|\frac{\partial f}{\partial L} dL\right| + \left|\frac{\partial f}{\partial R} dR\right|}{\left(\frac{L^2}{R^4}\right)} \cdot 100$$

$$= \frac{\frac{2L}{R^4}(.03L) + \left|\frac{-4L^2}{R^5}(.02R)\right|}{L^2/R^4} \cdot 100 = 6 + 8$$

(Note: each term on top has a $\frac{L^2}{R^4}$ factor)

$$= \boxed{14\%}$$

maximum error.

6. (6 points) Find the tangent plane in standard form for the graph of $x + 3y + 2z = 2\ln(x) + \ln(y) + \ln(z) + 6$ at the point $(1,1,1)$.

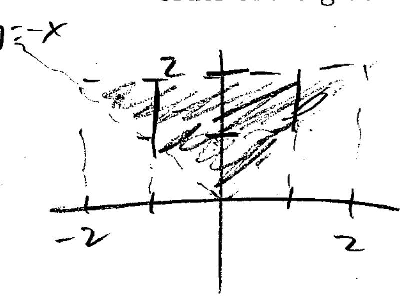
Let $f(x,y,z) = \ln(x^2yz) - x - 3y - 2z$. Then $\nabla f$ is $\perp$ to surface at $(1,1,1)$.

$$\nabla f(1,1,1) = \left\langle \frac{2}{x} - 1, \frac{1}{y} - 3, \frac{1}{z} - 2 \right\rangle\Big|_{(1,1,1)} = \langle 1, -2, -1 \rangle.$$

$\therefore$ the tangent plane's equation is

$$x - 2y - z = 1 - 2 - 1 \Rightarrow \boxed{x - 2y - z = -2}$$

7. (6 points) Sketch the region of integration for $I = \displaystyle\int_0^2 \int_{-y}^{y} f(x,y)\,dx\,dy$ and then rewrite $I$ with the order of integration switched to $dy\,dx$.




$$\Rightarrow \boxed{I = \int_{-2}^{0}\int_{-x}^{2} f(x,y)\,dy\,dx + \int_{0}^{2}\int_{x}^{2} f(x,y)\,dy\,dx}$$

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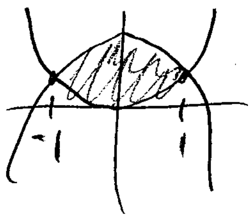
8. (8 points) Find the net volume for $f(x, y) = 5 - x^3 - yx$ over the region of integration bounded by $y = x^2$ and $y = 2 - x^2$.




$$\text{Net Vol} = \int_{-1}^{1}\int_{x^2}^{2-x^2} 5 - x^3 - yx \, dy\, dx$$

$$= \int_{-1}^{1} 5(2-x^2-x^2) - \underbrace{\cancel{x^3(2-2x^2)}}_{\text{ODD}}^{\to 0} - \underbrace{\cancel{\left(x \cdot \frac{y^2}{2}\Big|_{x^2}^{2-x^2}\right)}}_{\text{ODD}}^{\to 0} \, dx$$

$$= 2\left[10x - \frac{10x^3}{3}\right]_0^1 = 20\left(1 - \frac{1}{3}\right) = \boxed{\frac{40}{3}}$$

9. (8 points) Find the mass of the part of the solid sphere $x^2 + y^2 + z^2 \leq 9$ for which $x \geq 0$, $y \geq 0$, and $z \geq 0$ if the density is $\delta(x, y, z) = 4x \quad \text{kg/m}^3$.

$$\text{Mass} = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^3 4(\rho\sin\phi\cos\theta)\,\rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta \cdot \int_0^{\frac{\pi}{2}} \sin^2\phi \, d\phi \cdot \int_0^3 4\rho^3 \, d\rho$$

$$= \left(\sin\theta\Big|_0^{\frac{\pi}{2}}\right) \cdot \int_0^{\frac{\pi}{2}} \frac{1 - \cancel{\cos(2\phi)}^{\to 0}}{2} \, d\phi \cdot \left(\rho^4\Big|_0^3\right) \qquad \frac{\pi}{4}$$

$$= 1 \cdot \frac{\pi}{4} \cdot 81 = \boxed{\frac{81\pi}{4}}$$

T25

10. (8 points) Use Green's Theorem exactly once to evaluate $I = \int_C y - \cos(y)\,dx + x\sin(y)\,dy$ if $C$ is the circle $(x-3)^2 + (y+2)^2 = 16$ oriented clockwise.


$$I = -\int_{-C} \vec{F}\cdot d\vec{s}$$

$$\overset{G.T.}{=} -\iint_R \sin(y) - (1+\sin(y))\,dA$$

$$= -(-1)\iint_R dA$$

$$= \text{area circle}$$

$$= 16\pi$$

11. (7 points) Use Green's Theorem to calculate the area of the region inside the closed curve parameterized by $\vec{\alpha}(t) = \langle 2\cos(t) + \sin(t), \sin(t)\rangle$ for $0 \le t \le 2\pi$.

$$\text{Area} = \iint_R 1\,dA = \left|\oint_{\partial(R)} \langle 0, x\rangle \cdot d\vec{s}\right| \quad ; \; x = 2\cos(t) + \sin(t);$$

$$\vec{\alpha}'(t) = \langle -2\sin(t) + \cos(t), \cos(t)\rangle.$$

$$\therefore \text{Area} = \left|\int_0^{2\pi} \langle 0, 2\cos(t) + \sin(t)\rangle \cdot \vec{\alpha}'(t)\,dt\right|$$

$$= \left|\int_0^{2\pi} 2\cos^2(t) + \overbrace{\cancel{\sin(t)\cos(t)}}^{=\frac{\sin(2t)}{2};\,2\pi \to 0}\,dt\right|$$

$$= \int_0^{2\pi} 1 + \overset{\to 0;\,2\pi}{\cancel{\cos(2t)}}\,dt = 2\pi$$

12. (8 points) Use spherical coordinates to find the mass of the solid ~~in the first octant ($x$, $y$, and $z \geq 0$)~~ bounded by $z = \sqrt{x^2+y^2}$ and $z = \sqrt{2-x^2-y^2}$ if the mass is $\delta(x,y,z) = \cancel{24}\arctan\left(\frac{y}{x}\right)$.

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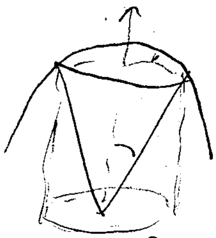
$x^2+y^2+z^2 = 2 \Rightarrow \rho = \sqrt{2}$



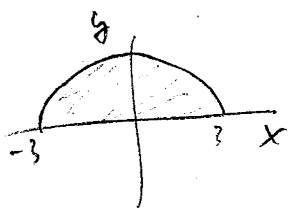
$$\text{Mass} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} 3\,\theta \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \theta\, d\theta \cdot \int_0^{\pi/4} \sin\phi\, d\phi \cdot \int_0^{\sqrt{2}} 3\rho^2\, d\rho$$

$$= \left(\frac{\theta^2}{2}\Big|_0^{2\pi}\right) \cdot \left(-\cos\phi\Big|_0^{\pi/4}\right) \cdot \rho^3\Big|_0^{\sqrt{2}}$$

$$= (2\pi^2)\left(1-\frac{\sqrt{2}}{2}\right)\cdot 2\sqrt{2} = \boxed{4\pi^2(\sqrt{2}-1)}$$

13. (6 points) Use cylindrical coordinates to evaluate $I = \int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} \frac{1}{\sqrt{x^2+y^2}}\, dz\, dy\, dx.$




$0 \leq r \leq 3$

$0 \leq \theta \leq \pi$

$0 \leq z \leq 9-r^2$

$\Rightarrow$ 

$$\therefore\ I = \int_0^{\pi}\int_0^{3}\int_0^{9-r^2} \frac{1}{r}\cdot r\, dz\, dr\, d\theta$$

$$= \int_0^{\pi} d\theta \cdot \int_0^{3} 9-r^2\, dr$$

$$= \pi \cdot \left(9r - \frac{r^3}{3}\Big|_0^3\right.$$

$$= \pi(27-9)$$

$$= 18\pi$$