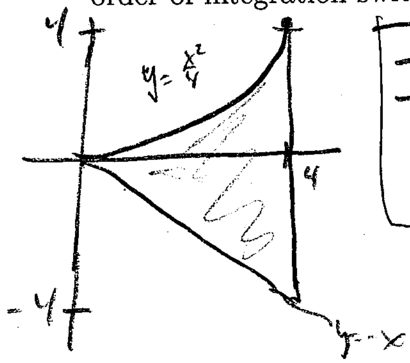
Test 2 100 points Math 200 Name: ____________________

1. (6 points) Sketch the region of integration for $I = \int_0^4 \int_{-x}^{x^2/4} f(x,y)\,dy\,dx$ and then rewrite I with the order of integration switched.



$$\boxed{I = \int_{-4}^{0}\int_{-y}^{4} f(x,y)\,dx\,dy + \int_{0}^{4}\int_{2\sqrt{y}}^{4} f(x,y)\,dx\,dy}$$

Sin(y) ✓

2. (7 points) Find the tangent plane in standard form for the graph of $ye^x + \tan(y)\cos(z) = z$ at the point $Q = (0, \pi, \pi)$.

$f(x,y,z) = ye^x + \sin(y)\cos(z) - z$

$\nabla f(0,\pi,\pi) = \langle ye^x, e^x + \cos(y)\cos(z), -\sin(y)\sin(z) - 1\rangle\Big|_{(0,\pi,\pi)}$

$= \langle \pi, 1 + (-1)(-1), -0\cdot 0 - 1\rangle = \underline{\langle \pi, 2, -1\rangle}$.

$\therefore\ \pi x + 2y - z = \langle \pi, 2, -1\rangle \cdot \langle 0, \pi, \pi\rangle \Rightarrow \boxed{\pi x + 2y - z = \pi}$

✓

3. (5 points) Approximate $f(3.2, 0.9)$ using a differential if $f(3,1) = 2$ and $\nabla f(3,1) = \langle 5, -2\rangle$.

$f(3.2, 0.9) \approx f(3,1) + 5(.2) - 2(-.1)$

$= 2 + 1 + .2 = \boxed{3.2}$

✓

4. (4 points) If $\vec{F}(x, y, \cancel{z}) = (mx^2y)\hat{\imath} + (x^3 - \cos(y))\hat{\jmath}$ then $\vec{F}$ is a gradient field if $m = 3$ and is not a gradient field if $m \neq 3$. Explain why this is true. Give a complete explanation.

$$\frac{\partial (x^3 - \cos(y))}{\partial x} = 3x^2 \quad ; \quad 3x^2 = mx^2 \Rightarrow \underline{m=3}.$$

$$\frac{\partial (mx^2y)}{\partial y} = mx^2$$

Also, the Domain of $\vec{F}$ is $\mathbb{R}^2$, a path connected, open and simply connected domain

$\Rightarrow m \neq 3 \Rightarrow \vec{F}$ is <u>not</u> a gradient field.

5. (7 points) Use the FTCLI to evaluate $I = \int_C 3x^2y\,dx + x^3 - \cos(y)\,dy$ if C is parameterized by $\vec{\alpha}(t) = \langle t\cos(t), t - 2\pi\rangle$ for $0 \leq t \leq \cancel{2\pi}$ π

$\phi(x,y) = x^3y - \sin(y)$ by guess/check.

Then $I = \phi(-\pi, -\pi) - \phi(0, -2\pi)$

$$= (\pi^4 - 0) - (0 - 0)$$

$$= \boxed{\pi^4}$$

6. (7 points) Evaluate $I = \int_C x^2y\,dx + x^3 - \cos(y)\,dy$ if C is parameterized by $\vec{\alpha}(t) = \langle t^2, t\rangle$ for $-\pi \leq t \leq \pi$.

$$I = \int_{-\pi}^{\pi} \langle t^5, t^6 - \cos(t)\rangle \cdot \langle 2t, 1\rangle \, dt$$

$$= \int_{-\pi}^{\pi} 2t^6 + t^6 - \cancel{\cos(t)}\, dt \quad 0; \pi$$

$$= 2 \cdot \frac{3t^7}{7}\Big|_0^{\pi} = \boxed{\frac{6}{7}\pi^7}$$

0; ODD and $x-y=-(y-x)$

7. (10 points) Evaluate $I=\int_{-1}^{1}\int_{0}^{y}\int_{y-x}^{x-y} \cancel{2z}+3\;dz\,dx\,dy$.

$$I=\int_{-1}^{1}\int_{0}^{y} 3\left[(x-y)-(y-x)\right]dx\,dy$$

$$=3\int_{-1}^{1}\int_{0}^{y} 2x-2y\;dx\,dy$$

$$=3\int_{-1}^{1} x^2-2yx\Big|_{0}^{y}\,dy$$

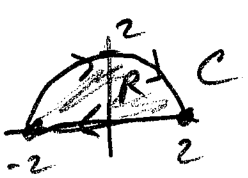
$$=3\int_{-1}^{1}-y^2\,dx=-2y^3\Big|_{0}^{1}=\boxed{-2}$$

8. (4 points) What is the tangent plane to the surface $z=f(x,y)$ at the point $P=(3,1,2)$ if $f(3,1)=2$ and $\nabla f(3,1)=\langle 4,5\rangle$? Write your final answer in $ax+by+cz=d$ form.

$$z-2=4(x-3)+5(y-1)$$

$$\Rightarrow\ \boxed{4x+5y-z=15}$$

9. (4 points) Use Green's Theorem and a double integral to evaluate $I=\int_C\langle\cos(x^3)-x^6,2x+e^{8y}\rangle\cdot d\vec{s}$ if C is the closed curve that starts at $(2,0)$ travels on the x-axis to ~~$(0,-2)$~~ $(-2,0)$ and then back to $(2,0)$ along the curve $y=\sqrt{4-x^2}$.



$$I\underset{G.T.}{=}-\iint_R 2-0\,dA=-2\,(\text{Area } R)$$

$$=-2\cdot\frac{\pi\cdot 2^2}{2}$$

$$=\boxed{-4\pi}$$

10. (8 points) Let $f(x,y) = x^2\cos(y)$, $x(u,w) = \dfrac{4u+w^2}{u+1}$, $y = y(w)$, $y(\cancel{3}) = \dfrac{\pi}{3}$, and $y'(\cancel{3}) = -\dfrac{2}{\sqrt{3}}$.

At $w=1$ and $u=0$

Find $\left.\dfrac{df}{dw}\right|_{\cancel{w=3}}$. Show organized work.

$\left.\begin{array}{l} w=1 \\ u=0 \end{array}\right\} \Rightarrow y = \frac{\pi}{3}$, $x = 1$

$$\begin{array}{c} f \\ x \quad y \\ u \; w \quad w \end{array} \Rightarrow \frac{\partial f}{\partial w} = f_x \cdot \frac{\partial x}{\partial w} + f_y \cdot \frac{dy}{dw}$$

$$= 2x\cos(y)\cdot\left(\frac{2w}{u+1}\right) + \left(-x^2\sin(y)\right)\cdot y'(w).$$

Then $f_w(0,1) = 2(1)\cos\left(\frac{\pi}{3}\right)\cdot 2 - 1^2\sin\left(\frac{\pi}{3}\right)\left(\frac{-2}{\sqrt{3}}\right)$

$= 4\cdot\frac{1}{2} + 1\cdot\left(\frac{\sqrt{3}}{2}\right)\left(\frac{2}{\sqrt{3}}\right) = 2+1 = \boxed{3}$

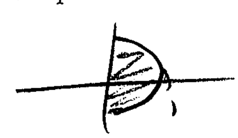
11. (4 points) Find the directional derivative of $f(x,y) = x\ln(x) - x\cos(2y)$ at the point $P = (e,\pi)$ in the direction of $\langle 1,1\rangle$.

$\nabla f = \langle \ln(x) + 1 - \cos(2y),\ +2x\sin(2y)\rangle$

$\Rightarrow \nabla f(P) = \langle 1+1-1,\ 2e\cdot 0\rangle = \langle 1,0\rangle.$ $\hat{u} = \dfrac{\langle 1,1\rangle}{\sqrt{2}}.$

Then $D_{\hat{u}}f(P) = \langle 1,0\rangle\cdot\dfrac{\langle 1,1\rangle}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}$

12. (6 points) Find the moment of inertia about the origin, I_0, for the lamina R equal to the half disk $0 \le x \le \sqrt{1-y^2}$ in the xy-plane if its density is $\delta(x,y) = \dfrac{x}{x^2+y^2}$.

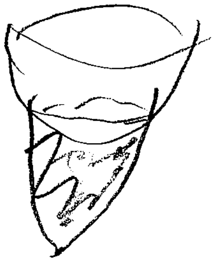


$$I_0 = \int_{-\pi/2}^{\pi/2}\int_0^1 \cancel{r^2}\cdot\frac{r\cos\theta}{\cancel{r^2}}\cdot r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta\cdot\int_0^1 r^2\,dr$$

$$= 2\left(\sin\theta\Big|_0^{\pi/2}\right)\cdot\left(\frac{r^3}{3}\Big|_0^1\right) = \boxed{\frac{2}{3}}$$

13. (7 points) Find the volume of the solid bounded by $z = x^2 + y^2$ and $z = 3x^2 + 3y^2 - 8$.



$$Vol = \int_0^{2\pi}\int_0^{2}\left[r^2 - (3r^2-8)\right] r\,dr\,d\theta$$

$$= 2\pi \cdot \left(-\frac{2r^4}{4} + \frac{8r^2}{2}\right|_0^2$$

$$= 2\pi(-8+16)$$

$$= \boxed{16\pi}$$

intersection:

$x^2+y^2 = 3x^2+3y^2-8$

$\Rightarrow 4 = x^2+y^2$

$2 = r$

14. (7 points) Use Green's Theorem to find the area of the region R bounded by the closed curve C parameterized by $\vec{p}(t) = \langle 5\cos(t) - \cos(5t), 5\sin(t) + 0.2\sin(5t)\rangle$ for $0 \le t \le 2\pi$.

$$\text{Area} = \iint_R 1\,dA \underset{G.T.}{=} \left|\oint_C \langle 0, x\rangle \cdot d\vec{s}\right|$$

$$\vec{p}\,'(t) = \langle -5\sin(t) + 5\sin(5t),\ 5\cos(t) + \cos(5t)\rangle$$

$$\therefore \text{Area} = \left|\int_0^{2\pi} 0 + (5\cos(t) - \cos(5t))\cdot(-5\cos(t) + \cos(5t))\,dt\right|$$

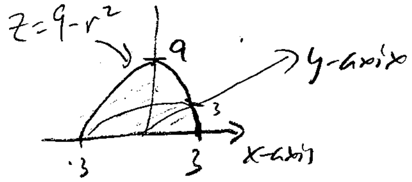
$$= \left|\int_0^{2\pi} 25\cos^2(t) - \cos^2(5t)\,dt\right|$$

$$= \left|\int_0^{2\pi} \frac{25}{2}\left(1 + \underbrace{\cancel{\cos(2t)}}_{0;\ 2\pi}\right) - \frac{1 + \underbrace{\cancel{\cos(10t)}}_{0;\ 10\pi}}{2}\,dt\right|$$

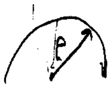
$$= 24\pi$$

15. let $I = \int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} \frac{1}{\sqrt{x^2+y^2}}\, dz\, dy\, dx$ for both parts below.

(a) (4 points) Write I using spherical coordinates. **Do not evaluate!** or Simplify



$$\boxed{I = \int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^{b} \frac{\rho^2 \sin\phi}{\rho \sin\phi}\, d\rho\, d\theta\, d\phi}$$



$\rho\cos\phi = 9 - \rho^2\sin^2\phi \Rightarrow \sin^2\phi\, \rho^2 + \cos\phi\, \rho - 9 = 0 \Rightarrow \rho = \frac{-\cos\phi \pm \sqrt{\cos^2\phi + 36\sin^2\phi}}{2\sin^2\phi}$; $\rho > 0$, use +.

(upper limit for ρ is +1 extra credit)

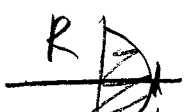
$\therefore b = \frac{-\cos\phi + \sqrt{\cos^2\phi + 36\sin^2\phi}}{2\sin^2\phi}$

Note: the limit as $\phi \to 0^+$ does equal 9.

(b) (4 points) Write I using cylindrical coordinates. **Do not evaluate!** or Simplify

$$\boxed{I = \int_0^{\pi}\int_0^{3}\int_0^{9-r^2} \frac{1}{r}\cdot r\, dz\, dr\, d\theta}$$

16. (6 points) Find the center of mass for the lamina R equal to the half disk $0 \le x \le \sqrt{1-y^2}$ in the xy-plane if its density is $\delta(x,y) = \frac{x}{x^2+y^2}$. $\bar{y}=0$ by symmetry, so just find $\bar{x}$.



~~R and δ are symmetrical about y=0, so $\bar{y}=0$, but I will also calculate it:~~

$M = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 \frac{r\cos\theta}{r^2}\cdot r\, dr\, d\theta = 2\sin\theta\Big|_0^{\frac{\pi}{2}} = 2.$

~~$M_x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 \frac{r^2\sin\theta\cos\theta}{r^2}\cdot r\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\theta\cos\theta\, d\theta \cdot \int_0^1 r\, dr = 0$~~ 0; odd

$M_y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 \frac{r^2\cos^2\theta}{r^2}\cdot r\, dr\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}\, d\theta \cdot \int_0^1 r\, dr = \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4}.$ 0; π

$\therefore (\bar{x},\bar{y}) = \left(\frac{\frac{\pi}{4}}{2}, \frac{0}{2}\right) = \boxed{\left(\frac{\pi}{8}, 0\right)}$